Příklad 39. *Vypočtěte integrál*

$$\iiint_A (x^2+y^2+z^2)\,\mathrm{d}A,$$

kde $A: x^2+y^2 \le a^2, z \ge 0, z \le b, a, b > 0$ *jsou parametry.*

válcové souřadnice

$x = r\cdot\cos\varphi$ $\quad r \in \mathbb{R}_0^+$

$y = r\cdot\sin\varphi$ $\quad 0 \le \varphi \le 2\pi$

$z = z$ $\quad z \in \mathbb{R}$

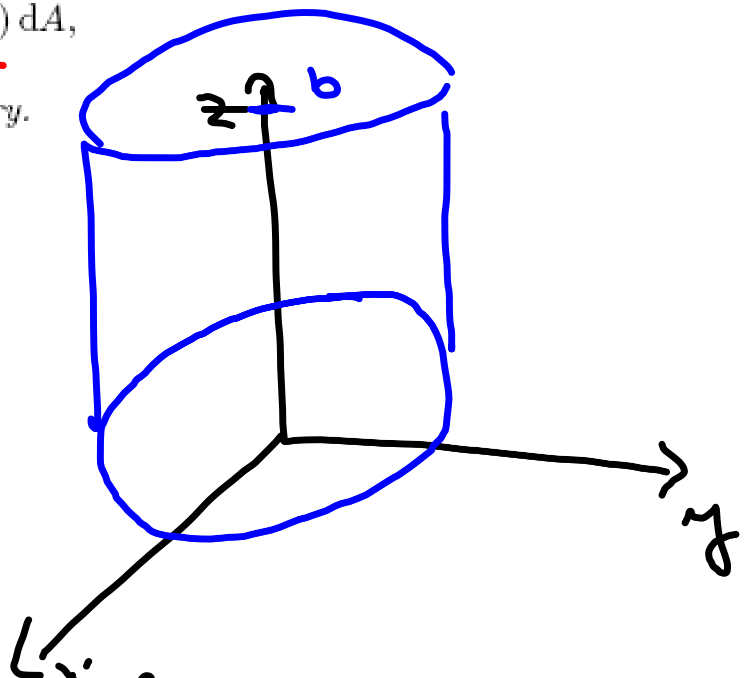
$J = D(\cdot) = r$

$r^2 \le a^2 \iff r \le a$

$0 \le z \le b$

$0 \le \varphi \le 2\pi$

$x^2+y^2+z^2 = r^2+z^2.$



$$\int_0^{2\pi}\int_0^b\int_0^a (r^2+z^2)\, r\, dr\, dz\, d\varphi = \int_0^{2\pi}\int_0^b \left[\frac{r^4}{4}+\frac{z^2r^2}{2}\right]_0^a dz\, d\varphi = \int_0^{2\pi}\int_0^b \left(\frac{a^4}{4}+\frac{z^2a^2}{2}\right) dz\, d\varphi =$$

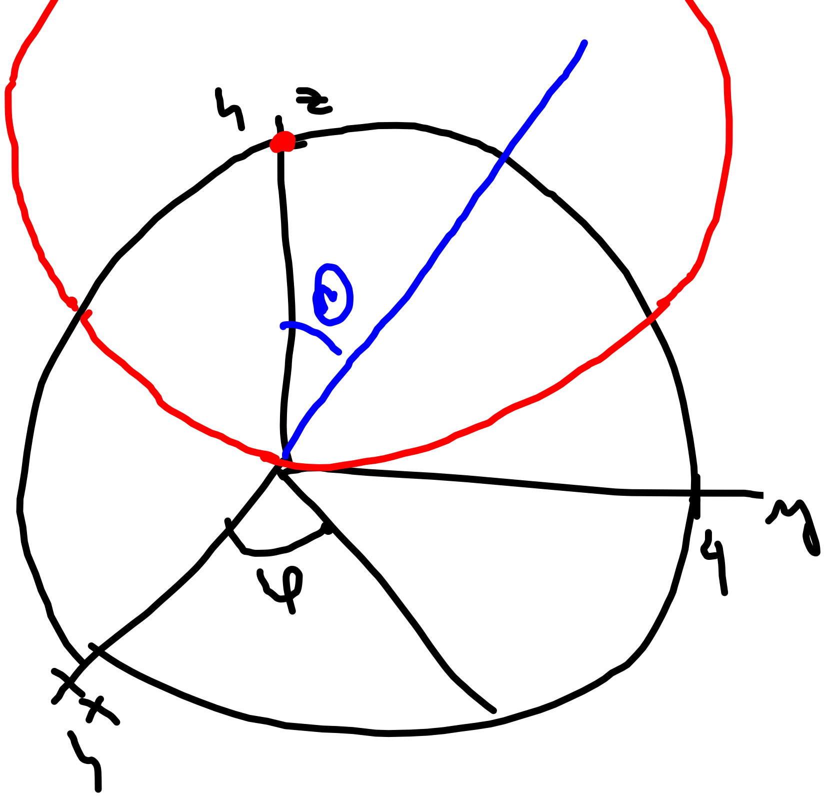
$$= 2\pi\cdot\left[\frac{a^4 z}{4}+\frac{a^2z^3}{6}\right]_0^b = 2\pi\left(\frac{a^4b}{4}+\frac{a^2b^3}{6}\right)$$

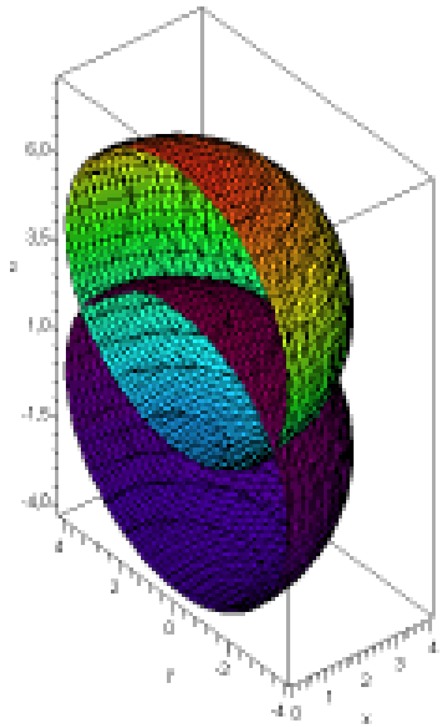
Příklad 40. *Vypočtěte objem množiny A, která je průnikem koulí $x^2+y^2+z^2 \le 16$, $x^2+y^2+z^2 \le 8z$*

(a) pomocí válcových souřadnic,

(b) pomocí sférických souřadnic.

$x^2+y^2+(z-4)^2 \le 16$





OBRÁZEK 28.

a) válcové souřadnice

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$
$J = r$

$0 \le \varphi \le 2\pi$

$0 \le r \le \sqrt{12}$

$r^2 + z^2 \le 16$
$r^2 + z^2 \le 8z \iff r^2 + (z-4)^2 \le 16$

$0 \le z \le 4$

(průnik sfér: $r^2 + z^2 = 16$
$r^2 + z^2 = 8z$

$16 - 8z = 0$
$z = 2$

$r^2 + 4 = 16 \Leftarrow$
$r^2 = 12$
$r = \sqrt{12}$)

fix r: $z^2 \le 16 - r^2$
$z \le \sqrt{16 - r^2}$

$(z-4)^2 \le 16 - r^2$ $\quad z \le 4$
$|z-4| \le \sqrt{16 - r^2} \iff z \ge 4 - \sqrt{16 - r^2}$

$$\int_0^{2\pi}\int_0^{\sqrt{12}}\int_{4-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r \, dz\, dr\, d\varphi = 2\pi\int_0^{\sqrt{12}} r\left(2\sqrt{16-r^2}-4\right) dr$$

$$= \left| \begin{array}{ll} r = 4\sin t & r=0 \Rightarrow t=0 \\ \sqrt{16-r^2} = \sqrt{16-16\sin^2 t} = |4\cos t| = 4\cos t & \\ dr = 4\cos t\, dt & r=\sqrt{12} \Rightarrow t = \arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3} \end{array} \right| =$$

$$= 2\pi \int_0^{\frac{\pi}{3}} 4\cdot\sin t\cdot(8\cos t - 4)\cdot 4\cos t\, dt =$$

$$= 2\pi \int_0^{\pi/3} (128\sin t\cos^2 t - 64\cos t\sin t)\, dt =$$

$$= 2\pi\left[128\cdot\frac{\cos^3 t}{-3} - 64\frac{\cos^2 t}{-2}\right]_0^{\frac{\pi}{3}} = 2\pi\left(-\frac{128}{3\cdot 8}+\frac{64}{2\cdot 4} - \left(-\frac{128}{3}+\frac{64}{2}\right)\right) = \frac{80}{3}\pi$$

b) $x = r\cos\varphi\sin\Theta$

$y = r\sin\varphi\sin\Theta$

$z = r\cos\Theta$

$J = r^2\sin\Theta$

$0 \le \varphi \le 2\pi$

$0 \le \Theta \le \frac{\pi}{2}$ $(\Leftarrow z \ge 0)$

$r\cos\Theta \ge 0$

$\cos\Theta \ge 0$

$\Theta \le \frac{\pi}{2}$

$x^2+y^2+z^2 \le 16$

$r^2 \le 16 \Leftrightarrow r \le 4$ – „horní" koule

$x^2+y^2+z^2 \le 8z$

$r^2 \le 8r\cos\Theta \Leftrightarrow r \le 8\cos\Theta$ – „dolní" koule

hraniční Θ: $z = 2 \Leftrightarrow r\cos\Theta = 2 \wedge r = 4 \Rightarrow \cos\Theta = \frac{1}{2}$

$\Rightarrow \Theta = \frac{\pi}{3}$

$\Rightarrow$ 2 integrály:

① $0 \le r \le 4$, $0 \le \Theta \le \frac{\pi}{3}$, $0 \le \varphi \le 2\pi$

② $0 \le r \le 8\cos\Theta$, $\frac{\pi}{3} \le \Theta \le \frac{\pi}{2}$, $0 \le \varphi \le 2\pi$

$$I_1 = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{4} r^2 \sin\Theta \, dr \, d\Theta \, d\varphi = 2\pi \cdot \int_0^{\pi/3} \sin\Theta \cdot \left[\frac{r^3}{3} \right]_0^4 d\Theta =$$

$$= 2\pi \cdot \int_0^{\pi/3} \frac{64}{3} \cdot \sin\Theta \, d\Theta = \frac{128}{3} \pi \cdot \left[-\cos\Theta \right]_0^{\pi/3} =$$

$$= \frac{128}{3} \pi \cdot \left(1 - \frac{1}{2} \right) = \frac{64}{3} \pi$$

$$I_2 = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{8\cos\Theta} r^2 \sin\Theta \, dr \, d\Theta \, d\varphi = 2\pi \cdot \int_{\pi/3}^{\pi/2} \sin\Theta \cdot \frac{1}{3} \cdot 8^3 \cos^3\Theta \, d\Theta$$

$$= \frac{2^{10}}{3} \cdot \pi \int_{\pi/3}^{\pi/2} \sin\Theta \cos^3\Theta \, d\Theta = \frac{2^{10}}{3} \pi \left[\frac{\cos^4\Theta}{-4} \right]_{\pi/3}^{\pi/2} =$$

$$= \frac{2^{10}}{3} \pi \cdot \frac{1}{2^6} = \frac{16}{3} \pi$$

$$I_1 + I_2 = \frac{80}{3} \pi .$$

Př.: $\displaystyle\int_0^{2\pi}\int_0^{1} x\,dx\,d\varphi =$

$$= \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \cdot \int_0^{1} x\,dx$$

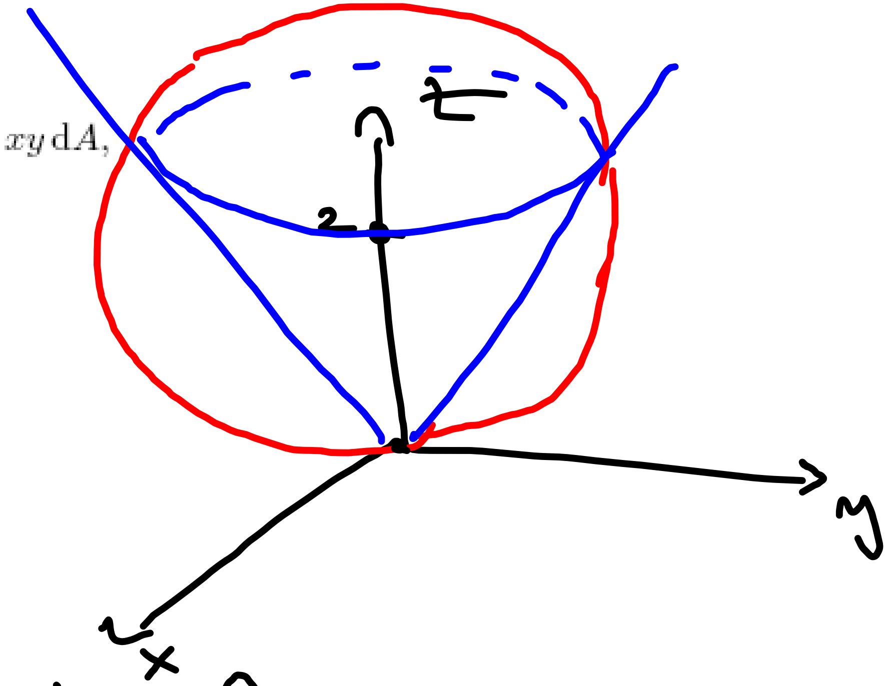
ALE!! $\displaystyle\int_0^{2\pi}\int_0^{\cos\varphi} x\,dx\,d\varphi =$

$$= \int_0^{2\pi} \frac{1}{2}\cos^2\varphi\,d\varphi \neq \int_0^{2\pi} d\varphi \cdot \int_0^{\cos\varphi} x\,dx$$

Příklad 41. *Vypočtěte integrál*

$$\iiint_A xy \, \mathrm{d}A,$$

kde $A: x^2 + y^2 + (z-2)^2 \leq 4, z \geq \sqrt{x^2+y^2}.$



$z^2 \geq x^2 + y^2$

střed koule:

$0 \leq \varphi \leq 2\pi, \quad 0 \leq \theta \leq \frac{\pi}{2}$

$r^2 \sin^2\theta + (r\cos\theta - 2)^2 \leq 4$

$r^2 \leq 4r\cos\theta \iff r \leq 4\cos\theta$

kužel: $r^2\cos^2\theta \geq r^2\sin^2\theta \iff \cos^2\theta \geq 1 - \cos^2\theta \iff \cos^2\theta \geq \frac{1}{2}$

$\iff \theta \leq \frac{\pi}{4}$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{4\cos\theta} r\cos\varphi\sin\theta \; r\sin\varphi\sin\theta \; r^2\sin\theta \, dr\, d\theta\, d\varphi =$$

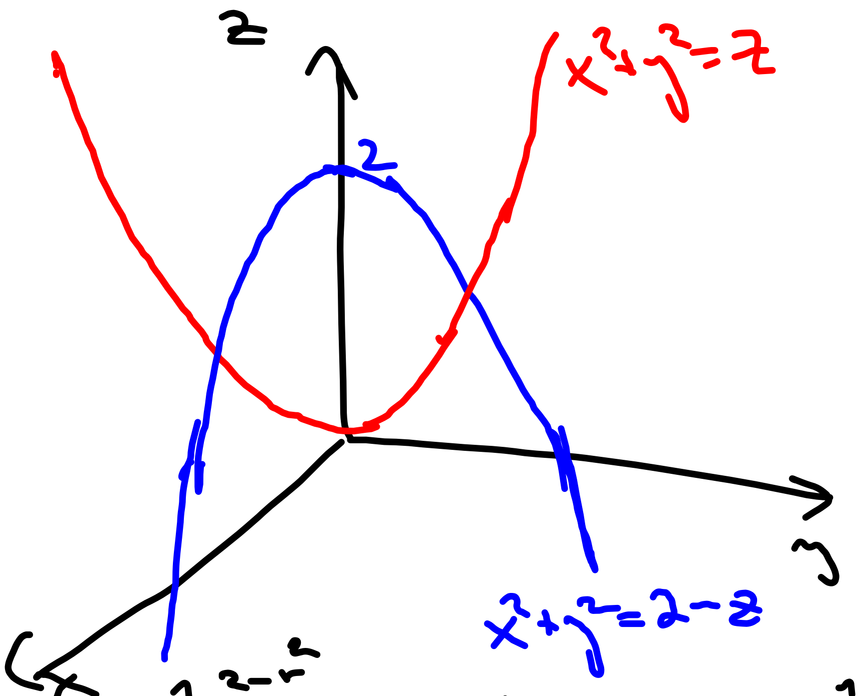
$$= \int_0^{2\pi} \cos\varphi\sin\varphi \, d\varphi \cdot \int \ldots = \left[\frac{\sin^2\varphi}{2}\right]_0^{2\pi} \cdot \ldots = 0$$

Příklad 42. *Určete souřadnice těžiště tělesa daného nerovnostmi*

$$x^2 + y^2 \leq z \leq 2 - (x^2 + y^2),$$

jehož hustota v daném bodě je trojnásobkem druhé mocniny vzdálenosti od roviny xy.

$\rho(x,y,z) = 3z^2$



$x_T = \frac{1}{M} \cdot \iiint x\,\rho(x,y,z)\,dx\,dy\,dz$

$y_T = \frac{1}{m} \cdot \iiint y\,\rho(x,y,z)\,dx\,dy\,dz$

$z_T = \frac{1}{m} \iiint z\,\rho(x,y,z)\,dx\,dy\,dz$

$M = \iiint \rho(x,y,z)\,dx\,dy\,dz$

válcové $x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$r^2 \leq z \leq 2 - r^2 \Rightarrow r^2 \leq 2 - r^2$

$\underline{r \leq 1}$

$M = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} 3z^2 \cdot r\,dz\,dr\,d\varphi = 2\pi \cdot \int_0^1 3r\left[\frac{z^3}{3}\right]_{r^2}^{2-r^2} =$

$$= 2\pi \int_0^1 \left(r\left((2-r^2)^3 - r^6\right)\right) dr =$$

$$= 2\pi \int_0^1 r\left(8 - 12r^2 + 6r^4 - r^6 - r^6\right) dr =$$

$$= 2\pi \left[4r^2 - 3r^4 + r^6 - \frac{r^8}{8} - \frac{r^8}{8}\right]_0^1 =$$

$$= 2\pi \cdot \frac{7}{4} = \boxed{\frac{7\pi}{2}}$$

$$x_T = \frac{1}{m} \cdot I_x, \text{ kde } I_x = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} \underline{r\cos\varphi} \cdot 3z^2 \cdot r \, dz\, dr\, d\varphi =$$

$$= \int_0^{2\pi} \cos\varphi \, d\varphi \cdot \iint \ldots = [\sin\varphi]_0^{2\pi} \cdot \ldots = 0$$

$$y_T = \frac{1}{m} \cdot I_y, \text{ kde } I_y = \ldots = \int_0^{2\pi} \sin\varphi \, d\varphi \ldots = [-\cos\varphi]_0^{2\pi} = 0$$

$$z_T = \frac{1}{m} \cdot I_z, \text{ kde } I_z = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} z \cdot 3z^2 \cdot r \, dz\, dr\, d\varphi =$$

$$= 2\pi \cdot \int_0^1 r \int_{r^2}^{2-r^2} 3z^3 \, dz \, dr =$$

$$= 2\pi \cdot \int_0^1 r \cdot \frac{3}{4} \left[z^4 \right]_{r^2}^{2-r^2} = 2\pi \cdot \frac{3}{4} \int_0^1 r \cdot \left[(2-r^2)^4 - r^8 \right] dr =$$

$$= \frac{3\pi}{2} \int_0^1 r \left(16 - 32r^2 + 24r^4 - 8r^6 + \cancel{r^8} - \cancel{r^8} \right) dr =$$

$$= \frac{3\pi}{2} \left[8r^2 - 8r^4 + 4r^6 - r^8 \right]_0^1 =$$

$$= \frac{3\pi}{2} \cdot 3 \Rightarrow z_T = \frac{1}{M} \cdot I_z = \frac{2}{7\pi} \cdot \frac{9\pi}{2} = \frac{9}{7}$$

Těžiště tělesa je v bodě $\left[0, 0, \frac{9}{7} \right]$