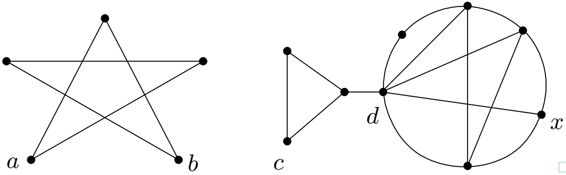
3 Distance in Graphs

While the previous lecture studied just the connectivity properties of a graph, now we are going to investigate how "long" (short, actually) a connection in a graph is.

This naturally leads to the concept of graph distance, which has two variants: the simple one considering only the number of edges, while the weighted one having a "length" for each edge.



Brief outline of this lecture

- Distance in a graph, basic properties, triangle inequality.
- Graph metrics: all-pairs shortest distances.
- Dijkstra's algorithm for the shortest weighted distance in a graph.
- Route planning: a sketch of some advanced ideas.

3.1 Graph distance (unweighted)

Recall that a walk of length n in a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ such that each e_i has the ends v_{i-1}, v_i.

Definition 3.1. **Distance** $d_G(u,v)$ between two vertices u, v of a graph G is defined as the length of the shortest walk between u and v in G.
If there is now walk between u, v, then we declare $d_G(u,v) = \infty$. □

Informally and naturally, the distance between u, v equals *the least possible number of edges* traversed from u to v. Specially $d_G(u,u) = 0$.

Recall, moreover, that the shortest walk is always a path – Theorem 2.2.

Fact: The distance in an undirected graph is symmetric, i.e. $d_G(u,v) = d_G(v,u)$. □

Lemma 3.2. *The graph distance satisfies the triangle inequality:*

$$\forall u, v, w \in V(G) \;\;:\;\; d_G(u,v) + d_G(v,w) \geq d_G(u,w)\,. \square$$

Proof. Easily; starting with a walk of length $d_G(u,v)$ from u to v, and appending a walk of length $d_G(v,w)$ from v to w, results in a walk of length $d_G(u,v) + d_G(v,w)$ from u to w. This is an upper bound on the real distance from u to w. □

How to find the distance

Theorem 3.3. *Let u, v, w be vertices of a connected graph G such that $d_G(u,v) < d_G(u,w)$. Then the breadth-first search algorithm on G, starting from u, finds the vertex v before w.* □

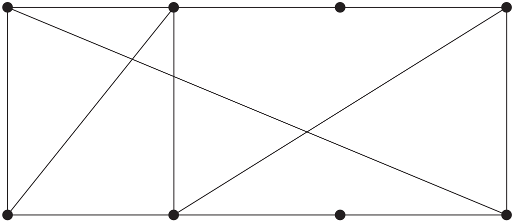
Proof. We apply induction on the distance $d_G(u,v)$: If $d_G(u,v) = 0$, i.e. $u = v$, then it is trivial that v is found first. So let $d_G(u,v) = d > 0$ and v' be a neighbour of v closer to u, which means $d_G(u,v') = d - 1$. Analogously choose w' a neighbour of w closer to u. Then

$$d_G(u,w') \geq d_G(u,w) - 1 > d_G(u,v) - 1 = d_G(u,v')\,,$$

and so v' has been found before w' by the inductive assumption. Hence v' has been stored into U before w', and (cf. FIFO) the neighbours of v' (v among them, but not w) are found before the neighbours of w' (such as w). □ □

Corollary 3.4. *The breadth-first search algorithm on G correctly determines graph distances from the starting vertex.*

Other related terms



Definition. Let G be a graph. We define, with respect to G, the following notions:

- The *excentricity* of a vertex $\text{exc}(v)$ is the largest distance from v to another vertex; $\text{exc}(v) = \max_{x \in V(G)} d_G(v, x)$. □
- The *diameter* $\text{diam}(G)$ of G is the largest excentricity over its vertices, and the *radius* $\text{rad}(G)$ of G is the smallest excentricity over its vertices. □
- The *center* of G is the subset $U \subseteq V(G)$ of vertices such that their excentricity equals $\text{rad}(G)$.

3.2 All-pairs shortest distances

Definition: The *metrics* of a graph is the collection of distances between all pairs of its vertices. In other words, the metrics is a matrix `d[,]` such that `d[i,j]` is the distance from i to j. □

Method 3.5. Dynamic programming for all-pairs distances
in a graph G on the vertex set $V(G) = \{v_0, v_1, \ldots, v_{N-1}\}$.

- Initially, let `d[i,j]` be 1 (alternatively, the edge length of $\{v_i, v_j\}$), or ∞ if v_i, v_j are not adjacent. □
- After step $t \geq 0$ let it hold that `d[i,j]` is the shortest length of a walk between v_i, v_j such that its internal vert. are from $\{v_0, v_1, \ldots, v_{t-1}\}$ (empty for $t = 0$).□
- Moving from step t to $t+1$, we update all the distances as:
  - Either `d[i,j]` from the previous step is still optimal (the vertex v_t does not help to obtain a shorter walk from v_i to v_j), or
  - there is a shorter v_i to v_j walk using (also) the vertex v_t which is, by the assumption at step t, of length `d[i,t]+d[t,j]`$\to$`d[i,j]`. □

Theorem 3.6. *Method 3.5 correctly computes the distance* `d[i,j]` *between each pair of vertices v_i, v_j in $N = |V(G)|$ steps.*

Remark: In a practical implementation we may use, say, `MAX_INT/2` in place of ∞.

Algorithm 3.7. Floyd–Warshall algorithm (cf. 3.5)

`input` < *the adjacency matrix* `G[,]` *of an* `N`*-vertex graph,*
such that the vertices of `G` *are indexed as* `0...N-1`*,*
and `G[i,j]=1` *if* i,j *adjacent and* `G[i,j]=0` *otherwise*;

```
for (i=0; i<N; i++) for (j=0; j<N; j++)
    d[i,j] = (i==j?0:  (G[i,j]?  1:  MAX_INT/2));
for (t=0; t<N; t++) {
    for (i=0; i<N; i++) for (j=0; j<N; j++)
        d[i,j] = min(d[i,j], d[i,t]+d[t,j]);
}
return 'The distance matrix d[,]'; □
```

Notice that this Algorithm 3.7 is extremely simple and relatively fast — it needs about N^3 steps to get the whole distance matrix.

Its only problem is that all-pairs distances must be computed at the same time, even if we need to know just one distance...

3.3 Weighted distance in graphs

Definition: A *weighted graph* is a graph G together with a weighting w of the edges by real numbers $w : E(G) \to \boldsymbol{R}$ (edge lengths in this case).
A *positively weighted graph* G, w is such that $w(e) > 0$ for all edges e. □

Definition 3.8. (Weighted distance) Consider a positively weighted graph G, w.
The length of the weighted walk $S = v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ in G is the sum

$$d_G^w(S) = w(e_1) + w(e_2) + \cdots + w(e_n)\,.$$

The *weighted distance* in G, w between a pair of vertices u, v is

$$d_G^w(u,v) = \min\{d_G^w(S) : S \text{ is a walk from } u \text{ to } v\}\,.$$ □

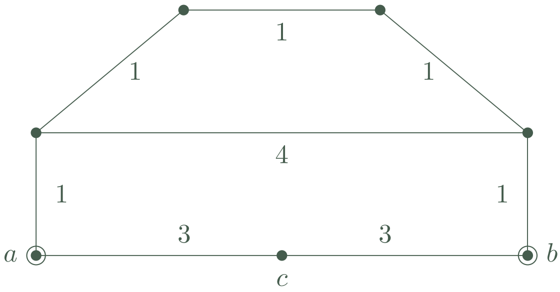
All these terms naturally extend from graphs to directed graphs. □

Analogously to Section 3.1 we get:

Fact: The shortest walk in a positively weighted (di)graph is always a path. □

Lemma 3.9. *The weighted distance in a positively weighted (di)graph satisfies the triangle inequality.*

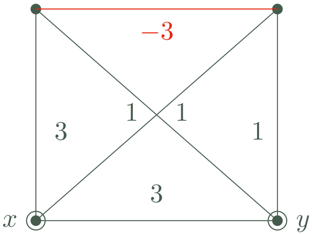
See an example...



The distances between a–c and between b–c are 3. What about the a–b distance? □
Is it 6? □ No, the distance from a to b in the graph is 5 (traverse the "upper path").

Negative edge-lengths?

What is the reason we are avoiding negative edge lengths?



Hence, what is the x–y distance this graph? Say, 3 or 1? □

No, it is $-\infty$, precisely by Definition 3.8, and this answer does not sound nice. . . □

Hence we have got a good reason not to consider negative edges in general.

3.4 Single-source shortest paths problem

This section deals with the more specific problem of finding the shortest distance between one pair of terminals in a graph (or, from a single source to all other vertices).

Remark: The coming Dijkstra's algorithm is, on one hand, slightly more involved than Algorithm 3.7, but it is significantly faster in the computation of *single-source shortest distances*, on the other hand. □

Dijkstra's algorithm:

- Is a variant of graph searching (related to BFS), in which every discovered vertex carries a *variable keeping its temporary distance*—the length of the shortest so far discovered walk reaching this vertex from the starting vertex. □
- We always pick from the depository the vertex with the shortest temporary distance. This is because no shorter walk may reach this vertex (assuming nonnegative edge lengths). □
- At the end of processing, the temporary distances become final shortest distances from the starting vertex (cf. Theorem 3.12).

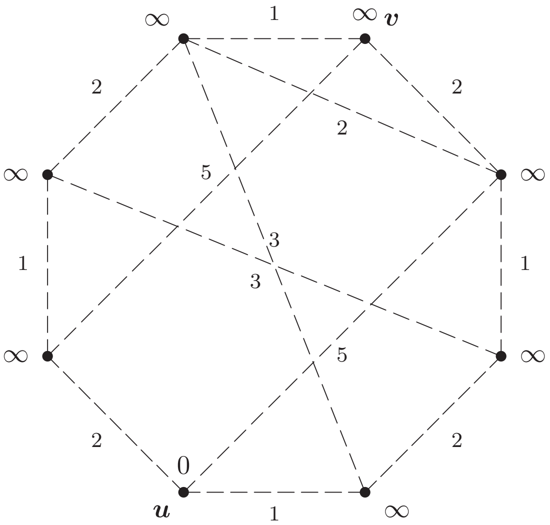
Algorithm 3.10. Computing the single-source shortest paths (Dijkstra),
i.e. finding the shortest walk from u to v, or from u to all other vertices.

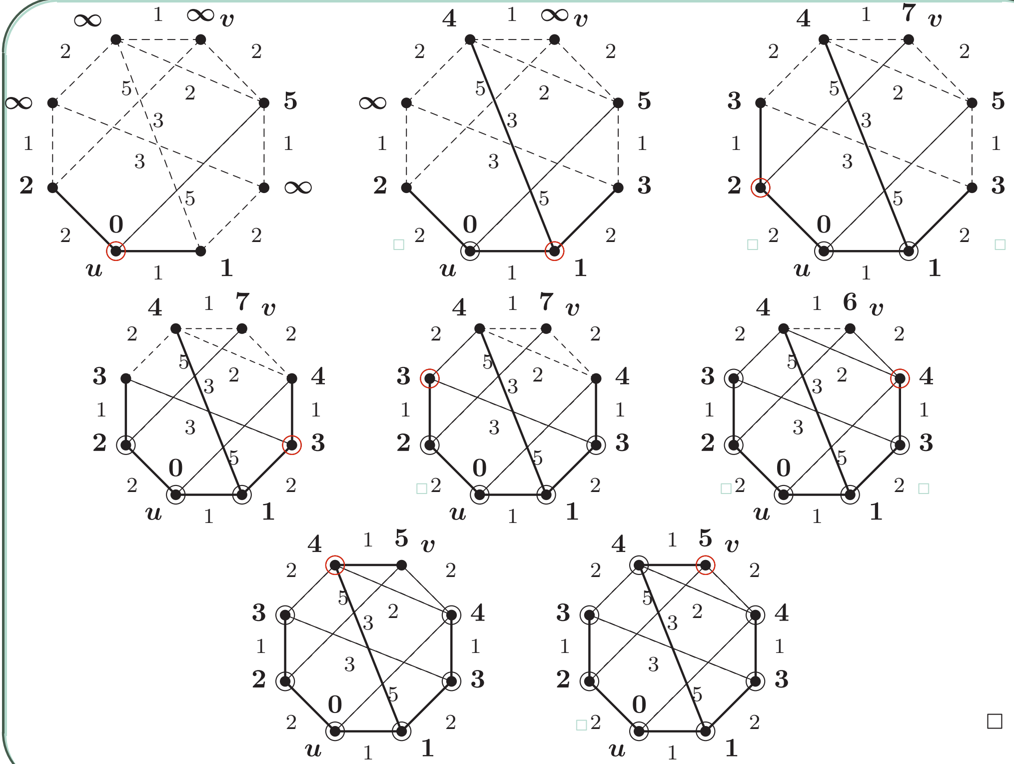
```
input < N-vertex graph given by adjacency mat. G[,] and cor. lengths len[,];
input < u,v, where u is the starting vertex and v the destination;□

// state[i] records the vertex processing state, dist[i] is the temporary distance
for (i=0; i<N; i++) { dist[i] = MAX_INT; state[i] = 'init'; }
dist[u] = 0;  depository D = {u};□
while (state[v]!='processed') {
    if (D==∅)  return 'No path';
    select m ∈ D with minimal dist[m];□
    // now updating all neighbours of m and their temporary distances
    for (i=0; i<N; i++) if (G[m,i]) {
        D = D ∪ {i};
        if (dist[m]+len[m,i]<dist[i]) {
            income[i] = m;
            dist[i] = dist[m]+len[m,i];
        }
    }
    state[m] = 'processed';  D = D \ {m};□
}
return 'A u-v path of length dist[v], stored in incm[] reversely';
```

Remark: Notice that Algorithm 3.10 works as-is also in directed graphs. □

Example 3.11. *An illustration run of Dijkstra's Algorithm 3.10 from u to v in the following graph.*




u
v

Fact: The number of steps performed by Algorithm 3.10 to find the shortest path from `u` to `v` is about N^2 when N is the number of vertices (not so good...). □
On the other hand, with a better implementation of the depository, one can achieve on sparse graphs runtime almost linear in the number of edges. □

Theorem 3.12. *Every iteration of Algorithm 3.10 (since just after finishing the first* `while()` *loop) maintains an invariant that*

- `dist[i]` *is the length of a shortest path from* `u` *to* `i` *using only those internal vertices* x *of* `state[x]=='processed'`. □

Proof: Briefly using *mathematical induction*:

- In the first iteration, the first vertex `m=u` is picked and processed, and its neighbours receive the correct straight distances (edge lengths). □

- In every next iteration, the picked vertex `m` is the nearest unprocessed one to the starting vertex `u`. Assuming nonnegative costs `len[,]`, this certifies that no shorter walk from `u` to `m` may exist in the graph. □

  On the other hand, any improved path from `u` to an unfinished vertex `i` passing through `m` has `mi` as the last edge (since the distance of `m` is not smaller than of the other finished vertices). Hence `dist[i]` is updated correctly in the algorithm.

□

3.5 Advanced route planning

In some situations, there is a better alternative to ordinary Dijkstra's algorithm — the *Algorithm A^** which uses a suitable *potential function* to direct the search "towards the destination". Whenever we have a good "sense of direction" (e.g. in a topo-map navigation), A^* can perform much better!

Algorithm A^*

- It re-implements Dijkstra with suitably modified edge costs. □
- Let $p_v(x)$ be a potential function giving an arbitrary lower bound on the distance from x to the destination v. E.g., in a map navigation, $p_v(x)$ may be the Euclidean distance from x to v. □
- Each directed(!) edge xy of the weighted graph G, w gets a new cost

$$\boldsymbol{w'(xy) = w(xy) + p_v(y) - p_v(x)}\,.$$

  The potential p_v is *admissible* when all $w'(xy) \geq 0$, i.e. $w(xy) \geq p_v(x) - p_v(y)$. The above Euclidean potential is always admissible. □
- The modified length of any u-v walk S then is $d_G^{w'}(S) = d_G^w(S) + p_v(v) - p_v(u)$, which is a constant difference from $d_G^w(S)$! Hence some S is optimal for the weighting w iff S is optimal for w'.
  Here the Euclidean potential "strongly prefers" edges in the dest. direction.