Příklad 1. *Zobrazte v rovině definiční obory funkcí:*

a) $f(x,y) = \ln(\frac{x}{3} + 2y)$,

b) $f(x,y) = \sqrt{1 - (\frac{x^2}{9} + \frac{y^2}{4})}$,

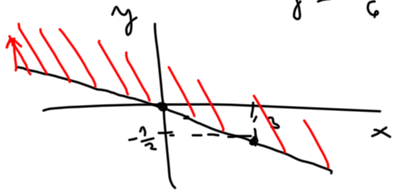
c) $f(x,y) = \frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$,

d) $f(x,y) = \arccos \frac{x}{x+y}$.

a) $\ln t \Rightarrow t > 0$

$\frac{x}{3} + 2y > 0$

$y > -\frac{x}{6}$



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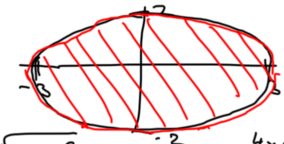
b) $f(x,y) = \sqrt{1 - (\frac{x^2}{9} + \frac{y^2}{4})}$ $\quad 1 \geq \frac{x^2}{9} + \frac{y^2}{4}$



c) $f(x,y) = \frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$

$x \geq \frac{y^2}{4} \Leftrightarrow 4x - y^2 \geq 0$

$1 - x^2 - y^2 > 0$

$x^2+y^2 < 1$ $\quad 1 - x^2 - y^2 \neq 1$

$x^2 + y^2 \neq 0$

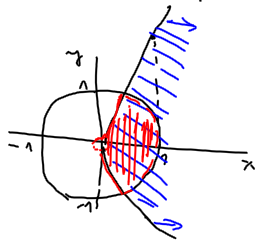
$\Leftrightarrow x \neq 0 \vee y \neq 0$

$x \geq \frac{y^2}{4}$

$y \geq 0 \Leftrightarrow \sqrt{x} \geq \frac{y}{2}$

$y \leq 2\sqrt{x}$

$y \leq 0 \Leftrightarrow \sqrt{x} \geq -\frac{y}{2}$



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d) $\arccos \frac{x}{x+y}$ je def. $\Leftrightarrow \frac{x}{x+y} \in \langle -1, 1 \rangle$

$\Leftrightarrow -1 \leq \frac{x}{x+y} \leq 1$

1. $x+y > 0$

$-(x+y) \leq x \leq x+y$

$-y \leq 2x \wedge 0 \leq y$

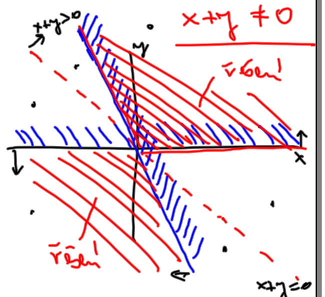
$y \geq -2x$

2. $x+y < 0$

$-(x+y) \geq x \geq x+y$

$y \leq -2x \wedge 0 \geq y$

$x+y \neq 0$



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Příklad 2. *Určete definiční obor funkcí:*

a) $f(x,y,z) = \sqrt{\frac{x}{y}} + \ln z$,

b) $f(x,y,z) = \arccos \frac{z}{\sqrt{x^2+y^2}}$.

a) $\frac{x}{y} \geq 0,\ y \neq 0,\ z > 0$

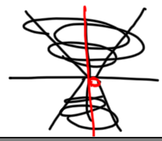
1. $y > 0,\ x \geq 0,\ z > 0$

2. $y < 0,\ x \leq 0,\ z > 0$

b) $-1 \leq \frac{z}{\sqrt{x^2+y^2}} \leq 1,\quad x^2+y^2 \neq 0,\quad x^2+y^2 \geq 0$ ✓

$x \neq 0 \vee y \neq 0$

$-\sqrt{x^2+y^2} \leq z \leq \sqrt{x^2+y^2}$



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Příklad 3. *Načrtněte vrstevnice funkcí:*

a) $f(x,y) = x^2 - y^2$,

b) $f(x,y) = x^y$, $x>0$

c) $f(x,y) = \sqrt{xy}$.

na hladině (úrovni) c:

$f_c = \{[x,y] \in \mathbb{R}^2;\ f(x,y) = c\}$

a) $c = x^2 - y^2$

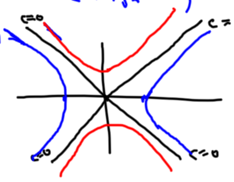
$y^2 = x^2 - c$

$c = 1$: $y \geq 0$

$y = \sqrt{x^2 - 1}$

$c = -1$: $y \geq 0$

$y = \sqrt{x^2 + 1}$



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b) $f(x,y) = x^y,\ x > 0$

$c = x^y = e^{y \cdot \ln x} \Rightarrow c > 0$

1. $c = 1$

$1 = e^{y \cdot \ln x}$

$0 = y \cdot \ln x$

$y = 0 \vee x = 1$

2. $c = e$

$e = e^{y \ln x}$

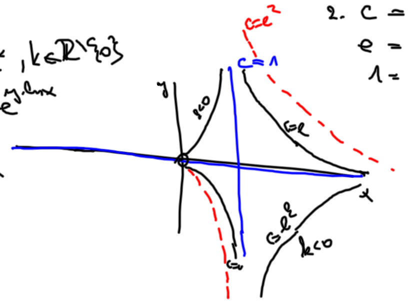
$1 = y \cdot \ln x$

$y = \frac{1}{\ln x}$

$c = e^k,\ k \in \mathbb{R} \setminus \{0\}$

$e^k = e^{y \cdot \ln x}$

$y = \frac{k}{\ln x}$



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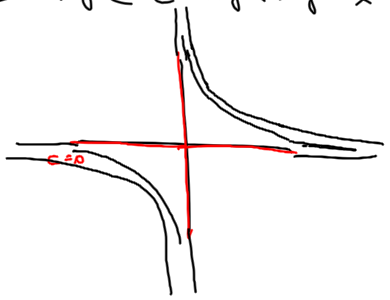
c) $f(x,y) = \sqrt{xy}$ $x \cdot y \geq 0$, $c \geq 0$

$c = \sqrt{xy} \Leftrightarrow c^2 = xy \Leftrightarrow y = \frac{c^2}{x}$ pro $x \neq 0$



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Příklad 4. *Rozhodněte, zda je daná posloupnost cauchyovská:*

a) $\left(-\frac{1}{2}\right)^n$, *kde* $n \in \mathbb{N}$,

b) a^n, *kde* $n \in \mathbb{N}$ *(v závislosti na hodnotě parametru* $a \in \mathbb{R}$*).*

c) $\frac{n+1}{n^2}$.

konvergentní $\Rightarrow$ cauchyovská $\{a_n\}$

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \forall m,n \geq N$

$|a_m - a_n| < \varepsilon$

a) $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \frac{1}{64}, \dots$

$N(\varepsilon)$: $\left|\left(-\frac{1}{2}\right)^m - \left(-\frac{1}{2}\right)^n\right| \leq 2 \cdot \frac{1}{2^m} \overset{?}{<} \varepsilon$

$m < n$

$2^{-m+1} < \varepsilon$

$-m+1 < \log_2 \varepsilon$

$m > 1 - \log_2 \varepsilon$

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b) a^n $n \in \mathbb{N}$ v závislosti na parametru $a \in \mathbb{R}$

- $a \in (-1,1)$ je cauchyovská – důkaz analogicky
- není cauchyovská ($a > 0 \Rightarrow$ roste nade všechny meze)

$a < 0$ osciluje

$a = -1$ $-1, 1, -1, 1, -1, 1$

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c) $\frac{n+1}{n^2}$ $n \in \mathbb{N}$

$a_n = \frac{1}{n} + \frac{1}{n^2} \longrightarrow 0$

konverguje $\Rightarrow$ cauchyovská

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Příklad 5. *Určete všechny hromadné body množiny:*

a) $\mathbb{Z}$,

b) $\mathbb{Q}$,

c) $\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \dots\}$,

d) $\{3(1 - \frac{1}{n}) + 2(-1)^n; n \in \mathbb{N}\}$.

a) nejsou

b) $\mathbb{R}$, $\mathbb{Q}$ je tzv. hustá v $\mathbb{R}$

c) $\mathbb{N}$

d) $3(1 - \frac{1}{n}) \longrightarrow 3$

podposl. pro $n = 2k$ $\longrightarrow 3+2$

– " – $n = 2k+1$ $\longrightarrow 3-2$

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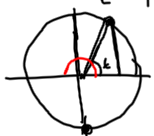
$f : \mathbb{R} \to \mathbb{R}^n$

Příklad 6. *Určete tečnu křivky dané předpisem*

$$f(t) = (2\cos t + \cos 3t, \sin 2t, t)$$

v bodě $t = \frac{3\pi}{2}$.

$f'(t) = (-2\sin t - 3\sin 3t, 2\cos 2t, 1)$ $[\cos t, \sin t]$



$f'\left(\frac{3\pi}{2}\right) = (2 - 3 \cdot 1, -2, 1)$

$= (-1, -2, 1)$

tečna: $\left(0, 0, \frac{3\pi}{2}\right) + \tau \cdot (-1, -2, 1)$, $\tau \in \mathbb{R}$

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$f(t)$

Příklad 7. *Na křivce* (t,t^2,t^3) *najděte takový bod, že jím procházející tečna je rovnoběžná s rovinou* $x+2y+z=1$.

normálový vektor roviny: $n=(1,2,1)$

$f'(t)=(1,2t,3t^2)$ $\quad f'(t)\perp n \Leftrightarrow f'(t)\cdot n = 0$

skal. součin

$1\cdot 1+2t\cdot 2+3t^2\cdot 1=0$

$3t^2+4t+1=0 \quad t_{1,2}=\frac{-4\pm\sqrt{16-12}}{6}=\begin{cases}-\frac{1}{3}\\ -1\end{cases}$

$t_1=-\frac{1}{3}\Rightarrow\left(-\frac{1}{3},\frac{1}{9},-\frac{1}{27}\right)$

$t_2=-1\Rightarrow(-1,1,-1)$

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Příklad 8. *Vypočtěte limity:*

a) $\lim_{(x,y)\to(1,1)}\frac{x+y}{\sqrt{x^2+y^2}}$,

b) $\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$,

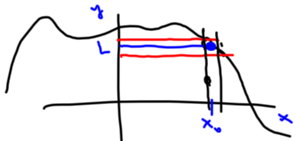
c) $\lim_{(x,y)\to(0,0)}xy^2\cos\frac{1}{xy^2}$,

d) $\lim_{(x,y)\to(0,0)}\frac{\sin xy}{x}$,

e) $\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y^2}$,

f) $\lim_{(x,y)\to(\infty,\infty)}(x^2+y^2)e^{-(x+y)}$,

g) $\lim_{(x,y)\to(\infty,1)}\left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}$.



$\lim_{x\to x_0} f(x)=L$

$\forall\varepsilon>0\ \exists\delta>0\ \forall|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$

a) $=f(1,1)=\frac{2}{\sqrt{2}}=\sqrt{2}$

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b) $\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}=$

$=\lim\frac{\cancel{(x^2+y^2)}(\sqrt{x^2+y^2+1}+1)}{\cancel{(x^2+y^2+1)-1}}=2$

c) $\lim_{(x,y)\to(0,0)}xy^2\cdot\boxed{\cos\frac{1}{xy^2}}=0$

$|\ \ |\le 1$ ohr.

$\boxed{\begin{array}{l}\lim f(x)\cdot g(x)\\ f(x)\to 0\\ |g(x)|\le K\in\mathbb{R}\end{array}}$

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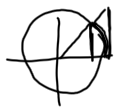
d) $\lim_{(x,y)\to(0,0)}\frac{\sin(xy)}{x}=$

$=\lim_{(x,y)\to(0,0)}\frac{\sin(xy)}{xy}\cdot y=$

$=\lim\frac{\sin(xy)}{xy}\cdot\lim y = 1\cdot 0=0$

$\lim_{t\to 0}\frac{\sin t}{t}=1$, $t=xy$, $\lim y \to 0$



e) $\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y^2}=$ vodorovně i svisle $\Rightarrow$ „asi 0"

pol. souřadnice: $x=r\cos\varphi$, $y=r\sin\varphi$ $\quad\lim_{r\to 0}\frac{r^3(\cos^3\varphi+\sin^3\varphi)}{r^2}=$

$=\lim_{r\to 0} r\,\underbrace{(\cos^3\varphi+\sin^3\varphi)}_{-2\le\dots\le 2}=0$

ohr.

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