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$$\underbrace{(1+x+x^2+x^3+\dots)}_{\text{jablka}}\underbrace{(1+x^2+x^4+x^6+\dots)}_{\text{banány}}\underbrace{(1+x^4+x^8+\dots)}_{\text{hrušky}}\underbrace{(1+x+x^2+x^3)}_{\text{pomeranče}}\underbrace{(1+x)}_{\text{pomelo}}$$

$$x^2 \cdot x^4 \cdot x^4 \cdot 1 \cdot 1 = x^{10}$$

(2J 4B 4H)

$$x^2 \cdot x^0 \cdot x^4 \cdot x^3 \cdot x = x^{10}$$

$$\vdots$$

$$\frac{1}{1-x} \cdot \frac{1}{\underset{1-x}{1-x^2}} \cdot \frac{1}{1-x^4} \cdot \frac{1-x^4}{1-x} \cdot (1+x) = \frac{1}{(1-x)^3}$$

Rozvineme:

$$(1-x)^{-3} = \binom{2}{2} + \binom{3}{2} \cdot x + \binom{4}{2} x^2 + \dots + \binom{n+3-1}{3-1} x^n + \dots$$

Odpověď: $\binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$

18 Rozviňte:

a) $$\frac{x}{x+2}=\frac{x}{2-(-x)}=\frac{x/2}{1-\left(-\frac{x}{2}\right)}=\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{8}-\frac{x^4}{16}+\dots+(-1)^{n}\frac{x^{n+1}}{2^{n+1}}+\dots$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{2^n}$$

obecněji: $a,b\in\mathbb{R}\setminus\{0\}$: $$\frac{1}{ax+b}=\frac{1}{b-(-ax)}=\frac{1/b}{1-\left(-\frac{ax}{b}\right)}=\sum_{n=0}^{\infty}\frac{1}{b}\cdot\left(-\frac{ax}{b}\right)^n$$

b) $$\frac{x^2+x+1}{2x^3-3x^2+1}=\frac{x^2+x+1}{(x-1)^2(2x+1)}=\frac{A}{2x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$2x^3-3x^2+1=(x-1)^2(2x+1)$$

	2	-3	0	1
1	2	-1	-1	0
1	2	1	0	

$$x^2+x+1=A(x-1)^2+B(2x+1)(x-1)+C(2x+1)$$

$x=1$: $3=3C \Rightarrow C=1$

$x=-\frac{1}{2}$: $\frac{3}{4}=A\left(-\frac{3}{2}\right)^2 \Rightarrow A=\frac{1}{3}$

$x=0$: $1=A-B+C \Rightarrow B=A+C-1=\frac{1}{3}$

$$\frac{x^2+x+1}{2x^3-3x^2+1}=\frac{1/3}{1+2x}-\frac{1/3}{1-x}+\frac{1}{(1-x)^2}=$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{3}(-2x)^n-\frac{1}{3}\cdot x^n+\binom{n+1}{1}x^n\right)=\sum_{n=0}^{\infty}\left[\frac{1}{3}\left((-2)^n-1\right)+(n+1)\right]x^n$$

$a_0, a_1, a_2, \ldots \longrightarrow a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots = A(x)$

$\alpha \cdot a_0, \alpha \cdot a_1, \ldots \longrightarrow \alpha \cdot (a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots) = \alpha \cdot A(x)$

$0, a_0, a_1, a_2 \longrightarrow 0 + a_0 x + a_1 x^2 + \ldots = x \cdot A(x)$

$a_1, a_2, \ldots \longrightarrow a_1 + a_2 x + a_3 x^2 = \dfrac{A(x) - a_0}{x}$

$a_0, 0, a_1, 0, a_2, 0, \ldots \longrightarrow a_0 + a_1 x^2 + a_2 x^4 + \ldots = A(x^2)$

79 $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \left(\dfrac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=1}^{\infty} n \cdot x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$

$\dfrac{1}{1-x} \xleftrightarrow{\text{v.f.p}} (1, 1, 1, \ldots)$ a) $\dfrac{1}{(1-x)^2} \xleftrightarrow{\text{v.f.p.}} (1, 2, 3, 4, \ldots)$

JINAK:

$\dfrac{x}{(1-x)^2} \longleftrightarrow (0, 1, 2, 3, 4, \ldots)$

$\left(\dfrac{x}{(1-x)^2}\right)' \longleftrightarrow (1 \cdot 1, 2 \cdot 2, 3 \cdot 3, \ldots)$

Obecněji: $f(x) \longleftrightarrow (a_0, a_1, \ldots)$ $f'(x) \xleftrightarrow{\text{v.f.p.}} (1 \cdot a_1, 2 \cdot a_2, 3 a_3, \ldots)$

b) $\cancel{f(x)}$ $\dfrac{1}{(1-x)^2} \xleftrightarrow{\text{v.f.p.}} (1, 2, 3, 4, \ldots)$

$\dfrac{2}{(1-x)^3} \xleftrightarrow{\text{v.f.p.}} (1 \cdot 2, 2 \cdot 3, 3 \cdot 4, \ldots) \Rightarrow f(x) = \dfrac{2}{(1-x)^3} - \dfrac{1}{(1-x)^2}$

$- \quad 1 \cdot 1, 2 \cdot 2, 3 \cdot 3$

$1 \cdot 1, 2 \cdot 1, 3 \cdot 1, \ldots$

c) $(1,1,2,2,4,4,8,8,\dots)$

$$\frac{1}{1-x} \longleftrightarrow (1,1,1,1,\dots)$$

$$A(x) = \frac{1}{1-2x} \longleftrightarrow (1,2,4,8,\dots)$$

$$A(x^2) = \frac{1}{1-2x^2} \longleftrightarrow (1,0,2,0,4,0,8,\dots)$$

$$x \cdot A(x^2) = \frac{x}{1-2x^2} \longleftrightarrow (0,1,0,2,0,4,0,8,\dots)$$

$$\frac{x+1}{1-2x^2} \longleftrightarrow (1,1,2,2,4,4,8,8,\dots)$$

d) $(9,0,0,2\cdot 16,0,0,4\cdot 25,0,0,8\cdot 36,\dots)$

$(9,2\cdot 16,4\cdot 25,8\cdot 36,\dots)$

$(9,16,25,36,\dots) = (3^2,4^2,5^2,\dots)$

$$f(x) = \frac{1+x}{(1-x)^3} \longleftrightarrow (1^2,2^2,3^2,\dots) \Longrightarrow \frac{f(x)-(1+4x)}{x^2} \longleftrightarrow (3^2,4^2,5^2,\dots)$$

subst. $x \leftarrow 2x \Rightarrow \dfrac{f(2x)-(1+8x)}{4x^2} \longleftrightarrow (1\cdot 3^2, 2\cdot 4^2, 4\cdot 5^2, 8\cdot 6^2,\dots)$

subst. $x \leftarrow x^3 \Rightarrow \dfrac{f(2x^3)-(1+8x^3)}{4x^6} \longleftrightarrow (9,0,0,2\cdot 16,0,0,\dots)$

e) $F(x) - x^2 \cdot F(x) + \frac{x}{1-x^3}$

$F(x) \longleftrightarrow (9, 0, 0, 32, 0, 0, \dots)$

$x^2 F(x) \longleftrightarrow (0, 0, 9, 0, 0, 32, \dots)$

$\frac{x}{1-x^3} \longleftrightarrow (0, 1, 0, 0, 1, 0, 0, 1, 0, \dots)$

(80) $a_n = a_{n-1} + 2a_{n-2} + (-1)^n + [n=1]$

$a_0 = \underbrace{a_{-1}}_{0} + \underbrace{2a_{-2}}_{0} + (-1)^0 = 1$

$a_1 = a_0 + \underbrace{2a_{-1}}_{0} + (-1)^{+1} = 0$

$* x^n$ a sečteme pro $n = 0 .. \infty$

$$\sum_{n=0}^{\infty} a_n x^n = x \cdot \sum_{n=0}^{\infty} a_{n-1} x^{n-1} + 2 \sum_{n=0}^{\infty} a_{n-2} x^n + \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} [n=1] x^n$$

$$\parallel$$

$$A(x) = x \cdot A(x) + 2x^2 \cdot A(x) + \frac{1}{1+x} + x$$

$$A(x)\left[1 - x - 2x^2\right] = \frac{1}{1+x} + x$$

$$A(x) = \frac{x^2 + x + 1}{(1+x)(1-x-2x^2)} = \frac{x^2 + x + 1}{(1+x)^2(1-2x)}$$

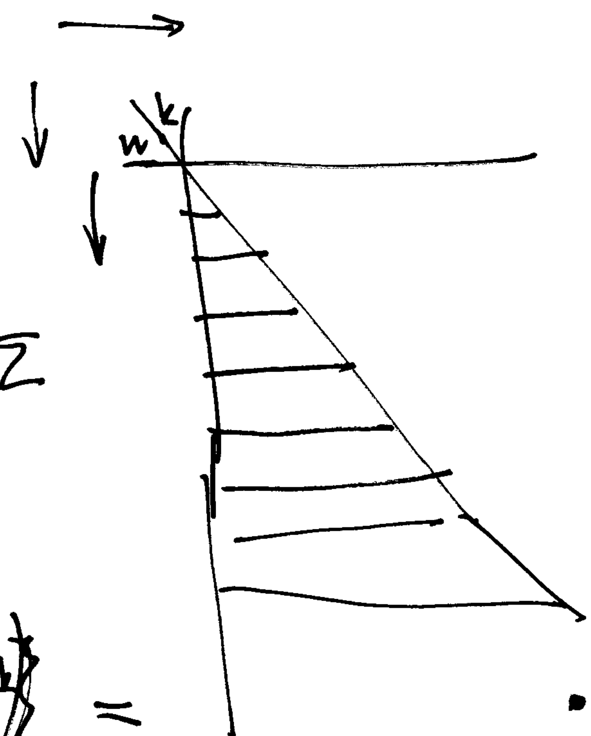
, dále anal. jako v (78)

(83) „pozorování v Př. 65" $1, 1, 2, 3, 5, 8, 13, 21$ $v_0 = 0$

Dokážeme indukcí, že $v_n = F_{2n}$: I $n=1$ $v_1 = 1 = F_2$

II. $v_n = v_{n-1} + \sum_{k=1}^{n-1} v_k + 1 \overset{IP}{=} F_{2n-2} + F_{2n-2} + F_{2n-4} + \dots F_4 + F_2 + F_1 = \dots = F_{2n-2} + \underbrace{F_{2n-2} + F_{2n-3}}_{F_{2n-1}} =$

$= F_{2n-2} + F_{2n-1} = F_{2n}$ $\square$



odvození bez „pozorování"

$V(x) \xleftrightarrow{v.f.p} v_n$, tj. $V(x) = \sum_{n=0}^{\infty} v_n x^n$

$$v_n = v_{n-1} + \sum_{k=1}^{n-1} v_k + 1 - [n=0] \quad \Big| \cdot x^n; \sum$$

$$V(x) = x \cdot V(x) + \sum_{n=0}^{\infty} \sum_{\substack{k=0 \\ k<n}}^{n-1} v_k x^n + \frac{1}{1-x} - 1$$

$$\sum_{k=0}^{\infty} \sum_{n>k} v_k x^n = \left(\sum_{k=0}^{\infty} v_k x^k\right) \cdot \left(\sum_{n>k} x^{n-k}\right) =$$

$$= \left(\sum_{k=0}^{\infty} v_k x^k\right) \cdot \frac{x}{1-x} = V(x) \cdot \frac{x}{1-x}$$

$$V(x) = x \cdot V(x) + \frac{x}{1-x} V(x) + \frac{x}{1-x} \Rightarrow \boxed{V(x) = \frac{x}{1-3x+x^2}} \xRightarrow{+82} v_n = F_{2n}$$