Part VIII

Elliptic curves cryptography and factorization

Cryptography based on manipulation of points of so called **elliptic curves** is currently getting momentum and has a tendency to replace public key cryptography based on the unfeasibility of factorization of integers, or on unfeasibility of the computation of discrete logarithms.

For example, the US-government has recommended to its governmental institutions to use mainly **elliptic curve cryptography.**

The main advantage of elliptic curves cryptography is that to achieve a certain level of security shorter keys are sufficient than in case of "usual cryptography". Using shorter keys can result in a considerable savings in hardware implementations.

The second advantage of the elliptic curves cryptography is that quite a few of attacks developed for cryptography based on factorization and discrete logarithm do not work for the elliptic curves cryptography.

It is amazing how practical is the elliptic curve cryptography that is based on very strangely looking theoretical concepts.

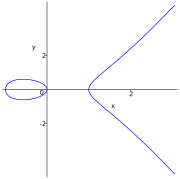
An elliptic curve E is the graph of the relation defined by the equation

$$E : y^2 = x^3 + ax + b$$

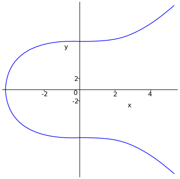
(where a, b are either rational numbers or integers (and computation is done modulo some integer n)) extended by a "point at infinity", denoted usually as ∞ (or 0) that can be regarded as being, at the same time, at the very top and very bottom of the y-axis.

We will consider mainly only those elliptic curves that have no multiple roots - what is equivalent to the condition $4a^3 + 27b^2 \neq 0$.

In case coefficients and x, y can be any rational numbers, a graph of an elliptic curve has one of the forms shown in the following figure. The graph depends on whether the polynomial $x^3 + ax + b$ has three or only one real root.



$y^2 = x(x+1)(x-1)$



$y^2 = x^3 + 73$

HISTORICAL REMARKS on ELLIPTIC CURVES

Elliptic curves are not ellipses and therefore it seems strange that they have such a name. Elliptic curves actually received their names from their relation to so called elliptic integrals

$$\int_{x1}^{x2} \frac{dx}{\sqrt{x^3+ax+b}} \qquad \int_{x1}^{x2} \frac{xdx}{\sqrt{x^3+ax+b}}$$

that arise in the computation of the arc-length of ellipses.

It may also seem puzzling why not to consider curves given by more general equations

$$y^2 + cxy + dy = x^3 + ex^2 + ax + b$$

The reason is that if we are working with rational coefficients or mod p, where $p > 3$ is a prime, then such a general equation can be transformed to our special case of equation. In other cases, it may be necessary to consider the most general form of equation.

ADDITION of POINTS on ELLIPTIC CURVES - GEOMETRY (1)

Geometry

On any elliptic curve we can define addition of points in such a way that points of the corresponding curve with such an operation of addition form an Abelian group. in which ∞ point is the identity element

If the line through two different points P_1 and P_2 of an elliptic curve E intersects E in a point $Q = (x, y)$, then we define $P_1 + P_2 = P_3 = (x, -y)$. (This also implies that for any point P on E it holds $P + \infty = P$.)

If the line through two different points P_1 and P_2 is parallel with y-axis, then we define $P_1 + P_2 = \infty$.

In case $P_1 = P_2$, and the tangent to E in P_1 intersects E in a point $Q = (x, y)$, then we define $P_1 + P_1 = (x, -y)$.

It should now be obvious how to define subtraction of two points of an elliptic curve.

It is now easy to verify that the above addition of points forms Abelian group with ∞ as the identity (null) element.

ELLIPTIC CURVES - GENERALITY

A general elliptic curve over Z_{p^m} where p is a prime is the set of points (x, y) satisfying so-called Weierstrass equation

$$y^2 + uxy + vy = x^3 + ax^2 + bx + c$$

for some constants u, v, a, b, c together with a single element $\mathbf{0}$, called the point of infinity.

If $p \neq 2$ Weierstrass equation can be simplified by transformation

$$y \rightarrow \frac{y - (ux + v)}{2}$$

to get the equation

$$y^2 = x^3 + dx^2 + ex + f$$

for some constants d, e, f and if $p \neq 3$ by transformation

$$x \rightarrow x - \frac{d}{3}$$

to get equation

$$y^2 = x^3 + fx + g$$

Formulas

Addition of points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ of an elliptic curve $E : y^2 = x^3 + ax + b$ can be easily computed using the following formulas:

$$P_1 + P_2 = P_3 = (x_3, y_3)$$

where

$$x_3 = \lambda^2 - x_1 - x_2$$
$$y_3 = \lambda(x_1 - x_3) - y_1$$

and

$$\lambda = \begin{cases} \dfrac{(y_2 - y_1)}{(x_2 - x_1)} & \text{if } P_1 \neq P_2, \\ \dfrac{(3x_1^2 + a)}{(2y_1)} & \text{if } P_1 = P_2. \end{cases}$$

All that holds for the case that λ is finite; otherwise $P_3 = \infty$.

Example For curve $y^2 = x^3 + 73$ and $P_1 = (2, 9)$, $P_2 = (3, 10)$ we have $P_1 + P_2 = P_3 = (-4, -3)$ and $P_3 + P_3 = (72, 611)$.

ELLIPTIC CURVES mod n

The points on an elliptic curve

$$E : y^2 = x^3 + ax + b \pmod n$$

are such pairs (x,y) mod n that satisfy the above equation, along with the point ∞ at infinity.

Example Elliptic curve $E : y^2 = x^3 + 2x + 3 \pmod 5$ has points

$$(1,1),(1,4),(2,0),(3,1),(3,4),(4,0),\infty.$$

Example For elliptic curve $E : y^2 = x^3 + x + 6(\mod 11)$ and its point $P = (2,7)$ holds $2P = (5,2)$; $3P = (8,3)$. Number of points on an elliptic curve (mod p) can be easily estimated.

Hasse's theorem If an elliptic curve $E(\bmod p)$ has N points then $|N - p - 1| < 2\sqrt{p}$

The addition of points on an elliptic curve mod n is done by the same formulas as given previously, except that instead of rational numbers c/d we deal with cd^{-1}

Example For the curve $E : y^2 = x^3 + 2x + 3$ it holds
$(1,4) + (3,1) = (2,0); (1,4) + (2,0) = (?,?)$.

ELLIPTIC CURVES DISCRETE LOGARITHM

Let E be an elliptic curve and A, B be its points such that $B = kA = (A + A + \dots A + A)$ – k times – for some k. The task to find such a k is called the discrete logarithm problem for elliptic curves.

No efficient algorithm to compute discrete logarithm problem for elliptic curves is known and also no good general attacks. Elliptic curves based cryptography is based on these facts.

There is the following general procedure for changing a discrete logarithm based cryptographic protocols to a cryptographic protocols based on elliptic curves:

- Assign to the message (plaintext) a point on an elliptic curve.
- Change, in the cryptographic protocol, modular multiplication to addition of points on an elliptic curve.
- Change, in the cryptographic protocol, exponentiation to multiplication of a point on the elliptic curve by an integer.
- To the point of an elliptic curve that results from such a protocol one assigns a message (cryptotext).

Problem and basic idea

The problem of assigning messages to points on elliptic curves is difficult because there are no polynomial-time algorithms to write down points of an arbitrary elliptic curve.

Fortunately, there is a fast randomized algorithm, to assign points of any elliptic curve to messages, that can fail with probability that can be made arbitrarily small.

Basic idea: Given an elliptic curve $E(\bmod p)$, the problem is that not to every x there is an y such that (x, y) is a point of E.

Given a message (number) m we therefore adjoin to m few bits at the end of m and adjust them until we get a number x such that $x^3 + ax + b$ is a square $\bmod p$.

Technicalities

Let K be a large integer such that a failure rate of $\frac{1}{2^K}$ is acceptable when trying to encode a message by a point.

For $j \in \{0, \dots, K\}$ verify whether for $x = mK + j$, $x^3 + ax + b \pmod p$ is a square (mod p) of an integer y.

If such an j is found, encoding is done; if not the algorithm fails (with probability $\frac{1}{2^K}$ because $x^3 + ax + b$ is a square approximately half of the time).

In order to recover the message m from the point (x, y), we compute:

$$\left\lfloor \frac{x}{K} \right\rfloor$$

ELLIPTIC CURVES KEY EXCHANGE

Elliptic curve version of the Diffie-Hellman key generation protocol goes as follows:

Let Alice and Bob agree on a prime p, on an elliptic curve E (mod p) and on a point P on E.

- Alice chooses an integer n_a, computes n_aP and sends it to Bob.
- Bob chooses an integer n_b, computes n_bP and sends it to Alice.
- Alice computes $n_a(n_bP)$ and Bob computes $n_b(n_aP)$. This way they have the same key.

ELLIPTIC CURVES VERSION of ElGamal CRYPTOSYSTEM

Standard version of ElGamal: Bob chooses a prime p, a generator $q < p$, an integer x, computes $y = q^x \pmod p$, makes public p, q, y and keeps x secret.

To send a message m Alice chooses a random r, computes:

$$a = q^r \ ; \ b = my^r$$

and sends it to Bob who decrypts by calculating $m = bx^{-x} \pmod p$

Elliptic curve version of ElGamal: Bob chooses a prime p, an elliptic curve E (mod p), a point P on E, an integer x, computes Q = xP, makes E, p, and Q public and keeps x secret.

To send a message m Alices expresses m as a point X on E, chooses random r, computes

$$a = rP \ ; \ b = X + rQ$$

And sends the pair (a, b) to Bob who decrypts by calculating $X = b - xa$.

ELLIPTIC CURVES DIGITAL SIGNATURES

Elliptic curves version of ElGamal digital signatures has the following form for signing (a message) m, an integer, by Alice and to have the signature verified by Bob:

Alice chooses p and an elliptic curve E (mod p), a point P on E and calculates the number of points n on E (mod p) – what can be done, and we assume that $0 < m < n$.

Alice then chooses a random integer a and computes Q = aP. She makes public p, E, P, Q and keeps secret a.

To sign m Alice does the following:

- Alice chooses a random integer $r, 1 \leq r < n$ such that gcd(r,n) = 1 and computes R = rP = (x,y).
- Alice computes $s = r^{-1}(m - ax) \pmod{n}$
- Alice sends the signed message (m,R,s) to Bob.

Bob verifies the signature as follows:

- Bob declares the signature as valid if xQ + sR = mP

  The verification procedure works because

$$xQ + sR = xaP + r^{-1}(m - ax)(rP) = xaP + (m - ax)P = mP$$

Warning Observe that actually $rr^{-1} = 1 + tn$ for some t. For the above verification procedure to work we then have to use the fact that $nP = \infty$ and therefore $P + t^{\infty} = P$

COMMENT

Federal (USA) elliptic curve digital signature standard (ECDSA) was introduced in 20??.

DOMAIN PARAMETERS for ELLIPTIC CURVES

To use ECC all parties involved have to agree on all basic elements concerning the elliptic curve E being used:

- A prime p.
- Constans a and b in the equation $y^2 = x^3 + ax + b$.
- Generator G of the underlying cyclic subgroup such that its order is prime.
- The order n of G, that is such an n that $nG = 0$
- Cofactor $h = \frac{|E|}{n}$ that should be small ($h \leq 4$) and, preferably $h = 1$.

To determine domain parameters (especially n and h) may be much time consuming task.

FACTORING with ELLIPTIC CURVES

Basis idea: To factorize an integer n choose an elliptic curve E, a point P on E and compute (modulo n) either iP for $i = 2, 3, 4, \ldots$ or $2^j P$ for $j = 1, 2, \ldots$. The point is that in doing that one needs to compute gcd(k,n) for various k. If one of these values is between 1 and n we have a factor of n.

Factoring of large integers: The above idea can be easily parallelised and converted to using an enormous number of computers to factor a single very large n. Each computer gets some number of elliptic curves and some points on them and multiplies these points by some integers according to the rule for addition of points. If one of computers encounters, during such a computation, a need to compute $1 < gcd(k, n) < n$, factorization is finished.

Example: If curve $E : y^2 = x^3 + 4x + 4 \pmod{2773}$ and its point $P = (1, 3)$ are used, then $2P = (1771, 705)$ and in order to compute 3P one has to compute $gcd(1770, 2773) = 59$ – factorization is done.

Example: For elliptic curve $E : y^2 = x^3 + x - 1 \pmod{35}$ and its point $P = (1, 1)$ we have $2P = (2, 2); 4P = (0, 22); 8P = (16, 19)$ and at the attempt to compute 9P one needs to compute $gcd(15, 35) = 5$ and factorization is done.

The only things that remain to be explored is how efficient is this method and when it is more efficient than other methods.

IMPORTANT OBSERVATIONS (1)

- If $n = pq$ for primes p, q, then an elliptic curve $E \pmod n$ can be seen as a pair of elliptic curves $E \pmod p$ and $E \pmod q$.
- It follows from the Lagrange theorem that for any elliptic curve $E \pmod n$ and its point P there is an $k < n$ such that $kP = \infty$.
- In case of an elliptic curve $E \pmod p$ for some prime p, the smallest positive integer m such that $mP = \infty$ for some point P divides the number N of points on the curve $E \pmod p$. Hence $NP = \infty$.

  If N is a product of small primes, then b! will be a multiple of N for a reasonable small b. Therefore, $b!P = \infty$.
- The number with only small factors is called smooth and if all factors are smaller than an b, then it is called b-smooth.

It can be shown that the density of smooth integers is so large that if we choose a random elliptic curve $E \pmod n$ then it is a reasonable chance that n is smooth.

PRACTICALITY of FACTORING USING ECC (1)

Let us continue to discuss the following key problem for factorization using elliptic curves:

Problem: How to choose k such that for a given point P we should try to compute points iP or $2^i P$ for all multiples of P smaller than kP?

Idea: If one searches for m-digits factors, one chooses k in such a way that k is a multiple of as many as possible of those m-digit numbers which do not have too large prime factors. In such a case one has a good chance that k is a multiple of the number of elements of the group of points of the elliptic curve modulo n.

Method 1: One chooses an integer B and takes as k the product of all maximal powers of primes smaller than B.

Example: In order to find a 6-digit factor one chooses B=147 and $k = 2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot \ldots \cdot 139$. The following table shows B and the number of elliptic curves one has to test:

PRACTICALITY of FACTORING USING ECC (2)

Digits of to-be-factors	6	9	12	18	24	30
B	147	682	2462	23462	162730	945922
Number of curves	10	24	55	231	833	2594

Computation time by the elliptic curves method depends on the size of factors.

ELLIPTIC CURVES FACTORIZATION - DETAILS

Given an n such that gcd(n, 6) = 1 and let the smallest factor of n be expected to be smaller than an F. One should then proceed as follows:

Choose an integer parameter r and:

1. Select, randomly, an elliptic curve

$$E : y^2 = x^3 + ax + b$$

   such that $gcd(n, 4a^2 + 27b^2) = 1$ and a random point P on E.

2. Choose integer bounds A,B,M such that

$$M = \prod_{j=1}^{l} p_j^{a_{p_j}}$$

   for some primes $p_1 < p_2 < \ldots < p_l \leq B$ and a_{p_j}, being the largest exponent such that $p_j^{a_j} \leq A$.

   Set j = k = 1

3. Calculate $p_j P$.

4. Computing gcd.
   - If $p_j P \neq O \pmod{n}$, then set $P = p_j P$ and reset $k \leftarrow k + 1$
     1. If $k \leq a_{p_j}$, then go to step (3).

ELLIPTIC CURVES FACTORIZATION - DETAILS II

2. If $k > a_{p_j}$, then reset $j \leftarrow j+1$, $k \leftarrow 1$.
   If $j \leq l$, then go to step (3); otherwise go to step (5)

- If $p_j P \equiv O(\mod n)$ and no factor of n was found at the computation of inverse elements, then go to step (5)

5. Reset $r \leftarrow r-1$. If $r > 0$ go to step (1); otherwise terminate with "failure".
   The "smoothness bound" B is recommended to be chosen as

$$B = e^{\sqrt{\frac{lnF(lnlnF)}{2}}}$$

and in such a case running time is

$$O(e^{\sqrt{2+o(1lnF(lnlnF))}} ln^2 n)$$

ELLIPTIC CURVES: FAQ

- How to choose (randomly) an elliptic curve E and point P on E? An easy way is first choose a point $P(x, y)$ and an a and then compute $b = y^2 - x^3 - ax$ to get the curve $E : y^2 = x^3 + ax + b$.
- What happens at the factorization using elliptic curve method, if for a chosen curve $E \pmod{n}$ the corresponding cubic polynomial $x^3 + ax + b$ has multiple roots (that is if $4a^3 + 27b^2 = 0$) ? No problem, method still works.
- What kind of elliptic curves are really used in cryptography? Elliptic curves over fields $GF(2^n)$ for $n > 150$. Dealing with such elliptic curves requires, however, slightly different rules.

FACTORIZATION

Factorization of integers is a very important problem.

A variety of techniques has been developed to deal with this problem.

So far the fastest classical factorization algorithms work in time

$$e^{O((\log n)^{\frac{1}{3}}(\log\log n)^{\frac{2}{3}})}$$

The fastest quantum algorithm for factorization works in (both quantum and classical) polynomial time.

In the rest of chapter several factorization methods will be presented and discussed.

FACTORIZATION on QUANTUM COMPUTERS

In the following we present the basic idea behind a polynomial time algorithm for quantum computers to factorize integers.

Quantum computers works with special quantum states on which are applied special unitary operation and very special quantum features are used.

Quantum computers work not with bits, that can take on any of two values 0 and 1, but with qubits (quantum bits) that can take on any of infnitely many states $\alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$.

REDUCTIONS

Shor's polynomial time quantum factorization algorithm is based on an understanding that factorization problem can be reduced

1. first on the problem of solving a simple modular quadratic equation;
2. second on the problem of finding period of functions $f(x) = a^x \bmod n$.

FIRST REDUCTION

Lemma If there is a polynomial time deterministic (randomized) [quantum] algorithm to find a nontrivial solution of the modular quadratic equations

$$a^2 \equiv 1 \pmod{n},$$

then there is a polynomial time deterministic (randomized) [quantum] algorithm to factorize integers.

Proof. Let $a \neq \pm 1$ be such that $a^2 \equiv 1 \pmod{n}$. Since

$$a^2 - 1 = (a+1)(a-1),$$

if n is not prime, then a prime factor of n has to be a prime factor of either $a+1$ or $a-1$. By using Euclid's algorithm to compute

$$gcd(a+1, n) \quad \text{and} \quad gcd(a-1, n)$$

we can find, in $O(\lg n)$ steps, a prime factor of n.

SECOND REDUCTION

The second key concept is that of the **period** of functions

$$f_{n,x}(k) = x^k \bmod n.$$

Period is the smallest integer r such that

$$f_{n,x}(k+r) = f_{n,x}(k)$$

for any k, i.e. the smallest r such that

$$x^r \equiv 1 \pmod{n}.$$

AN ALGORITHM TO SOLVE EQUATION $x^2 \equiv 1 \pmod{n}$.

1. *Choose randomly* $1 < a < n$.
2. *Compute* $gcd(a, n)$. *If* $gcd(a, n) \neq 1$ *we have a factor.*
3. *Find period* r *of function* $a^k \bmod n$.
4. *If* r *is odd or* $a^{r/2} \equiv \pm 1 \pmod{n}$,*then go to step 1; otherwise stop.*

If this algorithm stops, then $a^{r/2}$ is a non-trivial solution of the equation

$$x^2 \equiv 1 \pmod{n}.$$

EXAMPLE

Let $n = 15$. Select $a < 15$ such that $gcd(a, 15) = 1$.
$\{$The set of such a is $\{2, 4, 7, 8, 11, 13, 14\}\}$

Choose $a = 11$. Values of 11^x mod 15 are then

$$11, 1, 11, 1, 11, 1$$

what gives $r = 2$.

Hence $a^{r/2} = 11 \pmod{15}$. Therefore

$$gcd(15, 12) = 3, \qquad gcd(15, 10) = 5$$

For $a = 14$ we get again $r = 2$, but in this case

$$14^{2/2} \equiv -1 \pmod{15}$$

and the following algorithm fails.

1. *Choose randomly* $1 < a < n$.
2. *Compute* $gcd(a, n)$. *If* $gcd(a, n) \neq 1$ *we have a factor.*
3. *Find period* r *of function* $a^k \bmod n$.
4. *If* r *is odd or* $a^{r/2} \equiv \pm 1 \pmod{n}$,*then go to step 1; otherwise stop.*

EFFICIENCY of REDUCTION

Lemma If $1 < a < n$ satisfying $gcd(n, a) = 1$ is selected in the above algorithm randomly and n is not a power of prime, then

$$Pr\{r \text{ is even and } a^{r/2} \not\equiv \pm 1\} \geq \frac{9}{16}.$$

1. *Choose randomly* $1 < a < n$.
2. *Compute* $gcd(a, n)$. *If* $gcd(a, n) \neq 1$ *we have a factor.*
3. *Find period* r *of function* $a^k \bmod n$.
4. *If* r *is odd or* $a^{r/2} \equiv \pm 1 \pmod{n}$,*then go to step 1; otherwise stop.*

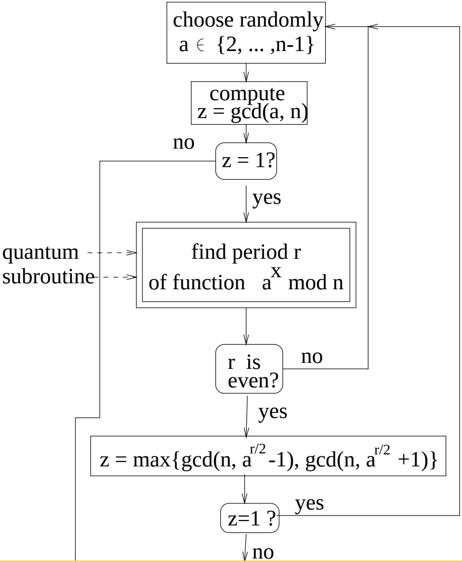
Corollary If there is a polynomial time randomized [quantum] algorithm to compute the period of the function

$$f_{n,a}(k) = a^k \bmod n,$$

then there is a polynomial time randomized [quantum] algorithm to find non-trivial solution of the equation $a^2 \equiv 1 \pmod{n}$ (and therefore also to factorize integers).

A GENERAL SCHEME for Shor's ALGORITHM

The following flow diagram shows the general scheme of Shor's quantum factorization algorithm



Factorization of so-called Fermat numbers $2^{2^i}+1$ is a good example to illustrate progress that has been made in the area of factorization.
Pierre de Fermat (1601-65) expected that all numbers

$$F_i = 2^{2^i} + 1 \qquad i \geq 1$$

are primes.

This is true for $i = 1, \dots, 4$. $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$.

1732 L. Euler found that $F_5 = 4294967297 = 641 \cdot 6700417$
1880 Landry+LeLasser found that

$$F_6 = 18446744073709551617 = 274177 \cdot 67280421310721$$

1970 Morrison+Brillhart found factorization for $F_7 = (39 digits)$

$$F_7 = 340282366920938463463374607431768211457 =$$
$$= 5704689200685129054721 \cdot 59649589127497217$$

1980 Brent+Pollard found factorization for F_8
1990 A. K. Lenstra+ ... found factorization for F_9 (155 digits)

Fermat TEST

It follows from the Little Fermat Theorem that if p is a prime, then for all $0 < b < p$, we have

$$b^{p-1} \equiv I \pmod{p}$$

Can we say that n is prime if and only if for all $0 < b < n$, we have

$$b^{n-1} \equiv I \pmod{n}?$$

No, there are composed numbers n, so-called Carmichael numbers, such that for all $0 < b < n$ that are co-prime with n it holds

$$b^{n-1} \equiv I \pmod{n}$$

Such number is, for example, n=561.

POLLARD ρ-METHOD

A variety of factorization algorithms, of complexity around $O(p^{\frac{1}{2}})$ where p is the smallest prime factor of n, is based on the following idea:

- A function f is taken that "behaves like a randomizing function" and $f(x) \equiv f(x \bmod p) \pmod p$ for any factor p of n – usually $f(x) = x^2 + 1$
- A random x_0 is taken and iteration

$$x_{i+1} = f(x_i) \mod n$$

  is performed (this modulo n computation actually "hides" modulo p computation in the following sense: if $x'_0 = x_0$, $x'_{i+1} = f(x'_i) \mod n$, then $x'_i = x_i \mod p$)
- Since $\mathbf{Z}_p$ is finite, the shape of the sequence x'_i will remind the letter ρ, with a tail and a loop. Since f is "random", the loop modulo n rarely synchronizes with the loop modulo p
- The loop is easy to detect by GCD-computations and it can be shown that the total length of tail and loop is $O(p^{\frac{1}{2}})$.

LOOP DETECTION

In order to detect the loop it is enough to perform the following computation:

$a \leftarrow x_0$; $b \leftarrow x_0$;
repeat
  $a \leftarrow f(a)$;
  $b \leftarrow f(f(b))$;
until a = b

Iteration ends if $a_t = b_{2t}$ for some t greater than the tail length and a multiple of the loop length.

FIRST Pollard ρ-ALGORITHM

Input: an integer n with a factor smaller than B
Complexity: $O(B^{\frac{1}{2}})$ of arithmetic operations

$x_0 \leftarrow random$; $a \leftarrow x_0$; $b \leftarrow x_0$;
do
  $a \leftarrow f(a) \mod n$;
  $b \leftarrow f(f(b) \mod n) \mod n$;
until gcd(a - b, n) $\neq 1$
output gcd(a - b, n)

The proof that complexity of the first Pollard factorization ρ-algorithm is given by $O(N^{\frac{1}{4}})$ arithmetic operations is based on the following result:
Lemma Let x_0 be random and f be "random" in Z_p, $x_{i+1} = f(x_i)$. The probability that all elements of the sequence

$$x_0, x_1, \ldots, x_t$$

are pairwise different when $t = 1 + \lfloor (2\lambda p)^{\frac{1}{2}} \rfloor$ is less than $e^{-\lambda}$.

SECOND Pollard ρ-ALGORITHM

Basic idea

1. Choose an easy to compute $f : Z_n \rightarrow Z_n$ and $x_0 \in Z_n$.

   Example $f(x) = x^2 + 1$

2. Keep computing $x_{i+1} = f(x_j)$, $j = 0, 1, 2, \ldots$ and $gcd(x_j - x_k, n)$, $k \leq j$. (Observe that if $x_j \equiv x_k \mod p$ for a prime factor p of n, then $gcd(x_j - x_k, n) \leq p$.)

   Example n $= 91$, $f(x) = x^2 + 1$, $x_0 = 1$, $x_1 = 2$, $x_2 = 5$, $x_3 = 26$

$$gcd(x_3 - x_2, n) = gcd(26 - 5, 91) = 7$$

Remark: In the ρ-method, it is important to choose a function f in such a way that f maps Z_n into Z_n in a "random" way.

Basic question: How good is the ρ-method?
(How long we expect to have to wait before we get two values x_j, x_k such that $gcd(x_j - x_k, n) \neq 1$, if n is not a prime?)

ρ-ALGORITHM

A simplification of the basic idea: For each k compute $gcd(x_k - x_j, n)$ for just one $j < k$.

Choose $f : Z_n \rightarrow Z_n, x_0$, compute $x_k = f(x_{k-1}), k > 0$.
If k is an (h +1)-bit integer, i.e. $2^h \leq k \leq 2^{h+1}$, then compute $gcd(x_k, x_{2^h-1})$.

Example n = 4087, $f(x) = x^2 + x + 1, x_0 = 2$

x_1 = f(2) = 7,	gcd($x_1 - x_0$, n) = 1
x_2 = f(7) = 57,	gcd($x_2 - x_1$, n) = gcd(57 – 7, n) = 1
x_3 = f(57) = 3307,	gcd($x_3 - x_1$, n) = gcd(3307 - 7, n) = 1
x_4 = f(3307) = 2745,	gcd($x_4 - x_3$, n) = gcd(2745 - 3307, n) = 1
x_5 = f(2746) = 1343,	gcd($x_5 - x_3$, n) = gcd(1343 - 3307, n) = 1
x_6 = f(1343) = 2626,	gcd($x_6 - x_3$, n) = gcd(2626 - 3307, n) = 1
x_7 = f(2626) = 3734,	gcd($x_7 - x_3$, n) = gcd(3734 - 3307, n) = 61

Disadvantage We likely will not detect the first case such that for some k_0 there is a $j_0 < k_0$ such that $gcd(x_{k0} - x_{j0}, n) > 1$.
This is no real problem! Let k_0 has h + 1 bits. Set $j = 2^{h+1} - 1, k = j + k_0 - j_0$. k has (h+2) bits, $gcd(x_k - x_j, n) > 1$

$$k < 2^{h+2} = 4 \cdot 2^h \leq 4k_0.$$

ρ-ALGORITHM

Theorem Let n be odd and composite and $1 < r < \sqrt{n}$ its factor. If f, x_0 are chosen randomly, then ρ algorithm reveals r in $O(\sqrt[4]{n} log^3 n)$ bit operations with high probability. More precisely, there is a constant $C > 0$ such that for any $\lambda > 0$, the probability that the ρ algorithm fails to find a nontrivial factor of n in $C\sqrt{\lambda}\sqrt[4]{n} log^3 n$ bit operations is less than $e^{-\lambda}$.

Proof Let C_1 be a constant such that gcd(y - z, n) can be computed in $C_1 log^3 n$ bit operations whenever $y, z < n$.
Let C_2 be a constant such that f(x) mod n can be computed in $C_2 log^2 n$ bit operations if $x < n$.
If k_0 is the first index for which there exists $j_0 < k_0$ with $x_{k0} \equiv x_{j0} \mod r$, then the ρ-algorithm finds r in $k \leq 4k_0$ steps.
The total number of bit operations is bounded by $\rightarrow 4k_0(C_1 log^3 n + C_2 log^2 n)$
By Lemma the probability that k_0 is greater than $1 + \sqrt{2\lambda r}$ is less than $e^{-\lambda}$.
If $k_0 \leq 1 + \sqrt{2\lambda r}$, then the number of bit operations needed to find r is bounded by

$$4(1 + \sqrt{2\lambda r})(C_1 log^3 n - C_2 log^2 n) < 4(1 + \sqrt{2\lambda}\sqrt[4]{n})(C_1 log^3 n + C_2 log^2 n)$$

If we choose $C > 4\sqrt{2}(C_1 + C_2)$, then we have that r will be found in $C\sqrt{\lambda}\sqrt[4]{n} log^3 n$ bit operations – unless we made uniform choice of (f, x_0) the probability of what is at most $e^{-\lambda}$.

COMMENTS

Pollard ρ-method works fine for integers n with a small factor.

Next method, so called Pollard (p-1)-method, works fine for n having a prime factor p such that all prime factors of p-1 are small.

When all prime factors of p-1 are smaller than a B, we say that p-1 is B-smooth.

POLLARD's p-1 algorithm

Pollard's algorithm (to factor n given a bound b on factors).

a := 2;

for j=2 **to** b **do** $a := a^j \mod n$;

$f := gcd(a-1, n)$; $\quad f = gcd(2^{b!} - 1, n)$

if $1 < f < n$ **then** f is a factor of n **otherwise** failure

Indeed, let p be a prime divisor of n and $q < b$ for every prime $q|(p-1)$.
(Hence $(p-1)|b!$).

At the end of the **for**-loop we have

$$a \equiv 2^{b!} \pmod n$$

and therefore

$$a \equiv 2^{b!} \pmod p$$

By Fermat theorem $2^{p-1} \equiv 1 \pmod p$ and since $(p-1)|b!$ we get $a \equiv 2^{b!} \equiv 1 \pmod p$.
and therefore we have $p|(a-1)$
Hence

$$p|gcd(a-1, n)$$

IMPORTANT OBSERVATIONS II

Pollard ρ-method works fine for numbers with a small factor.

The p-1 method requires that p-1 is smooth. The elliptic curve method requires only that there are enough smooth integers near p and so at least one of randomly chosen integers near p is smooth.

This means that the elliptic curves factorization method succeeds much more often than p-1 method.

Fermat factorization and Quadratic Sieve method discussed later works fine if integer has two factors of almost the same size.

Fermat FACTORIZATION I

If n = pq, $p < \sqrt{n}$, then

$$n = \left(\frac{q+p}{2}\right)^2 - \left(\frac{q-p}{2}\right)^2 = a^2 - b^2$$

Therefore, in order to find a factor of n, we need only to investigate the values

$$x = a^2 - n$$

$$\text{for } a = \left\lceil\sqrt{n}\right\rceil + 1, \left\lceil\sqrt{n}\right\rceil + 2, \ldots, \frac{(n-1)}{2}$$

until a perfect square is found.

Basic idea: Factorization is easy if one finds x, y such that $n|(x^2 - y^2)$

Proof: If n divides (x + y)(x - y) and n does not divide neither x+y nor x-y, then one factor of n has to divide x+y and another one x-y.

Example $n = 7429 = 227^2 - 210^2$, $x = 227$, $y = 210$
$x - y = 17$ $x + y = 437$
$\gcd(17, 7429) = 17$ $\gcd(437, 7429) = 437$.

How to find such x and y?

First idea: one tries all t starting with $\sqrt{n}$ until $t^2 - n$ is a square S^2.

Second idea: One forms a system of (modular) linear equations and determines x and y from the solutions of such a system.

number of digits of n	50	60	70	80	90	100	110	120
number of equations	3000	4000	7400	15000	30000	51000	120000	245000

METHOD of QUADRATIC SIEVE to FACTORIZE an INTEGER n

Step 1 One finds numbers x such that $x^2 - n$ is small and has small factors.

Example
$$\left.\begin{array}{l} 83^2 - 7429 = -540 = (-1) \cdot 2^2 \cdot 3^3 \cdot 5 \\ 87^2 - 7429 = 140 = 2^2 \cdot 5 \cdot 7 \\ 88^2 - 7429 = 315 = 3^2 \cdot 5 \cdot 7 \end{array}\right\} \text{relations}$$

Step 2 One multiplies some of the relations if their product is a square.
For example

$$(87^2 - 7429)(88^2 - 7429) = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 = 210^2$$

Now

$$(87 \cdot 88)^2 \equiv (87^2 - 7429)(88^2 - 7429) \mod 7429$$
$$227^2 \equiv 210^2 \mod 7429$$

Hence 7429 divides $227^2 - 210^2$.
Formation of equations: For the i-th relation one takes a variable λ_i and forms the expression
$((-1) \cdot 2^2 \cdot 3^3 \cdot 5)^{\lambda_1} \cdot (2^2 \cdot 5 \cdot 7)^{\lambda_2} \cdot (3^2 \cdot 5 \cdot 7)^{\lambda_3} = (-1)^{\lambda_1} \cdot 2^{2\lambda_1 + 2\lambda_2} \cdot 3^{2\lambda_1 + 2\lambda_2} \cdot 5^{\lambda_1 + \lambda_2 + \lambda_3} \cdot 7^{\lambda_2 + \lambda_3}$

If this is to form a quadrat the following equations have to hold

$$\begin{aligned} \lambda_1 &\equiv 0 \mod 2 \\ \lambda_1 + \lambda_2 + \lambda_3 &\equiv 0 \mod 2 \\ \lambda_2 + \lambda_3 &\equiv 0 \mod 2 \\ \lambda_1 = 0, \quad & \lambda_2 = \lambda_3 = 1 \end{aligned}$$

METHOD of QUADRATIC SIEVE to FACTORIZE n

Problem How to find relations?
Using the algorithm called Quadratic sieve method.

Step 1 One chooses a set of primes that can be factors – a so-called factor basis.

One chooses an m such that $m^2 - n$ is small and considers numbers $(m+u)^2 - n$ for $-k \leq u \leq k$ for small k.
One then tries to factor all $(m+u)^2 - n$ with primes from the factor basis, from the smallest to the largest.

u	-3	-2	-1	0	1	2	3
$(m+u)^2 - n$	-540	-373	-204	-33	140	315	492
Sieve with 2	-135		-51		35		123
Sieve with 3	-5		-17	-11		35	41
Sieve with 5	-1				7	7	
Sieve with 7					1	1	

In order to factor a 129-digit number from the RSA challenge they used

8 424 486 relations
569 466 equations
544 939 elements in the factor base

APPENDIX to CHAPTER 8

HISTORY of ELLIPTIC CURVES CRYPTOGRAPHY

- The use of elliptic curves in cryptography was suggested independently by Neal Koblitz and Victor S. Miller in 1985.
- Behind is a believe that the discrete logarithm of a random elliptic curve element with respect to publically known base point is infeasible.
- At first Elliptic curves over a prime finite field were used for ECC. Later also elliptic curves ovf the fiels $GF(2^m)$ started to be used.
- In 2005 the US NSA endorsed to use ECC (Elliptic curves cryptography) with 384-bit key to protect information classified as "top secret".
- There are patents in force covering certain aspects of ECC technology.
- Elliptic curves have been first used for factorization by Lenstra.
- Eliptic curves played an important role in perhaps most celebrated mathematical proof of the last hundred years - in the proof of Fermat's last Twwwwi - due to A. Wiles and R. Taylor.

SECURITY of ELLIPTIC CURVE CRYPTOGRAPHY

- Security of ECC depends on the difficulty of solving the discrete logarithm problem over elliptic curves.
- Two general methods of solving such a discrete logarithm problem are known.
- The square root method and Silver-Pohling-Hellman (SPH) method.
- SPH method factors the order of a curve into small primes and solves the discrete logarithm problem as a combination of discrete logarithms for small numbers.
- Computation time of the square root method is proportional to $O(\sqrt{e^n})$ where n is the order of the based element of the curve.

FACTORIZATION of a 512-BIT NUMBER

On August 22, 1999, a team of scientists from 6 countries found, after 7 months of computing, using 300 very fast SGI and SUN workstations and Pentium II, factors of the so-called RSA-155 number with 512 bits (about 155 digits).

RSA-155 was a number from a Challenge list issue by the US company RSA Data Security and "represented" 95 % of 512-bit numbers used as the key to protect electronic commerce and financial transmissions on Internet.

Factorization of RSA-155 would require in total 37 years of computing time on a single computer.

When in 1977 Rivest and his colleagues challenged the world to factor RSA-129, he estimated that, using knowledge of that time, factorization of RSA-129 would require 10^{16} years.

LARGE NUMBERS

Hindus named many large numbers – one having 153 digits.

Romans initially had no terms for numbers larger than 10^4.

Greeks had a popular belief that no number is larger than the total count of sand grains needed to fill the universe.

Large numbers with special names:

googol - 10^{100} googolplex - $10^{10^{100}}$

FACTORIZATION of very large NUMBERS

W. Keller factorized F_{23471} which has 10^{7000} digits.

J. Harley factorized: $10^{10^{1000}} + 1$.

One factor: 316,912,650,057,350,374,175,801,344,000,001

1992 E. Crandal, Doenias proved, using a computer that F_{22}, which has more than million of digits, is composite (but no factor of F_{22} is known).

Number $10^{10^{10^{34}}}$ was used to develop a theory of the distribution of prime numbers.