

9 About Intersection Graphs

Since this lecture we focus on selected detailed topics in Graph theory that are “close to your teacher’s heart”...

The first selected topic is that of *intersection graphs*, i.e. of graphs that are defined by the intersecting pairs of certain – often geometric – objects. This area of graphs is motivated both by its illustrative nature and by its practical applicability (e.g. *interval graphs*).

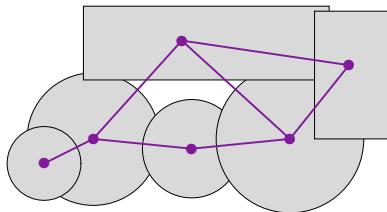


Brief outline of this lecture

- What are intersection graphs; the interval graphs as an example.
- Chordal graphs and their properties.
- Several more commonly studied intersection and geometric classes.
- String and segment representations of graphs as another example.

9.1 Intersection graphs; Interval graphs

Definition 9.1. The **intersection graph** of a set family \mathcal{M} is the graph $I_{\mathcal{M}}$ on the vertices $V = \mathcal{M}$ and edges $E = \{\{A, B\} \subset \mathcal{M} : A \cap B \neq \emptyset\}$.



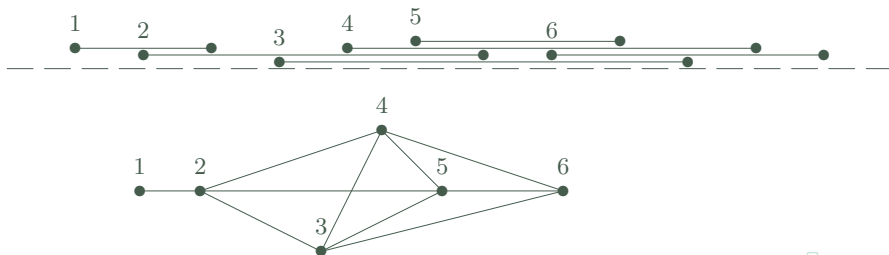
We remark that typical examples of intersection graphs are of geometric nature. \square

Fact: Specific intersection graph classes are always closed on induced subgraphs. \square

Proposition 9.2. *Every simple graph is isomorphic to the intersection graph of a suitable set system.*

Interval graphs

One of the oldest studied examples of intersection graphs are the *interval graphs* (shortly *INT*) – the intersection graphs of intervals on a line.

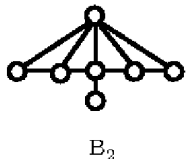
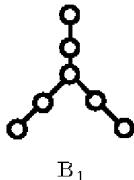
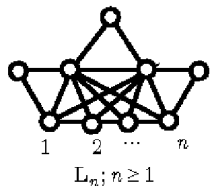
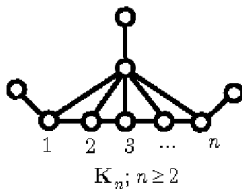
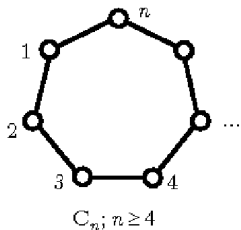


Recall that these graphs have already been implicitly used in connection with the single job assignment problem, which actually was the colouring problem for interval graphs.

Lemma 9.3. *Every cycle of length more than three in an interval graph has a **chord**.*

Theorem 9.4. The class of interval graphs has the following characterizations: □

- A graph is *INT* if and only if it has *no induced subgraph* isomorphic to one of



□

- A simple graph G is *INT* if and only if G has *no induced C_4* , and the complement of G has a *transitive orientation*.

9.2 Chordal graphs

Definition: A graph G is *chordal* if there exists **no induced** cycle (i.e. no chordless cycle) of length > 3 in G .

For example:



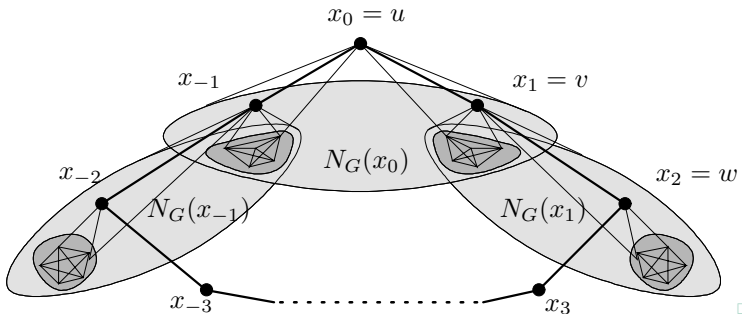
□

Theorem 9.5. Every chordal graph G contains a *simplicial vertex*, which is a vertex s such that the neighbours of s in G form a *clique*. □

Proof: We moreover say that a graph H is *bisimplicial* if H is complete, or H contains two nonadjacent simplicial vertices. Then...

The proof is accomplished in the following **tricky sequence** of three relatively straightforward claims:

1. It holds that for every cycle C in any chordal graph G , and an edge e there exists an edge f in G such that $E(C) \setminus \{e\} \cup \{f\}$ contains a triangle. \square
2. Let $e = uv$ be an edge of G and let $N_G(v)$ – the neighbours of v – induce a bisimplicial subgr. of G . If v is simplicial in $N_G(u)$ but not in whole G , then there is another w adjacent to v but not to u , such that w is simplicial in $N_G(v)$.



3. Hence if G is **not bisimplicial**, but the neighbourhoods of its vertices all induce bisimplicial subgraphs, then G contains a **cycle C contradicting (1)**. \square
4. Therefore, G is bisimplicial. \square

Simplicial decomposition

From Theorem 9.5, one can easily conclude:

Corollary 9.6. *Every chordal graphs has a **simplicial decomposition**, i.e. a vertex ordering $V(G) = (v_1, v_2, \dots, v_n)$ such that each v_i , $i = 2, \dots, n$, is simplicial in the subgraph induced on the vertex subset $\{v_1, \dots, v_{i-1}\}$. \square*

Fact: Simplicial decompositions can be used to build efficient recognition algorithms for chordal and interval graphs.

Another intersection characterization

The following shows that chordal graphs actually present a natural generalization of interval graphs. . .

Theorem 9.7. *A graph G is chordal if and only if there exists a tree T such that G is the intersection graph of a collection of subtrees in T . \square*

Proof (only a sketch of \implies); by induction on the number of vertices of G :

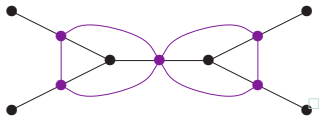
- This is trivial for one vertex. \square
- Let v be a simplicial vertex of G , and let $G_0 = G - v$. Then G_0 has an intersection representation by subtrees in a tree T_0 . \square

The neighbours of v in G form a clique $K \subseteq G$, and all the trees representing vertices of K must intersect in a joint node $x \in V(T_0)$. We construct T by adding a new leaf y in T_0 adjacent to x , and represent the vertex v by a tree $\{y\}$. \square

9.3 More intersection graph examples

We briefly and informally introduce few more commonly studied types (classes) of intersection graphs, mostly of geometric nature.

- A *line graph* $L(G)$ of a graph G is the intersection graph of the edges $E(G)$.



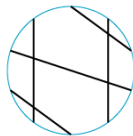
- *Circular-interval graphs* (CA) are the intersection graphs of intervals on a circle. □
- *Circle graphs* (CIR) are the intersection graphs of straight chords of a circle.



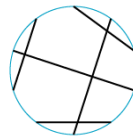
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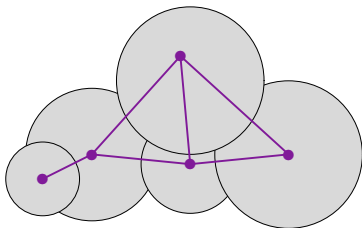
C_5



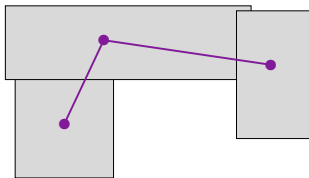
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- *Disc graphs* (DISC) are the intersection graphs of closed discs in the plane. Furthermore, unit-disc graphs are such that all the discs have unit size.



- *Box graphs* (BOX) are the intersection graphs of axis-parallel “boxes” (from rectangles to higher dimensional bodies).



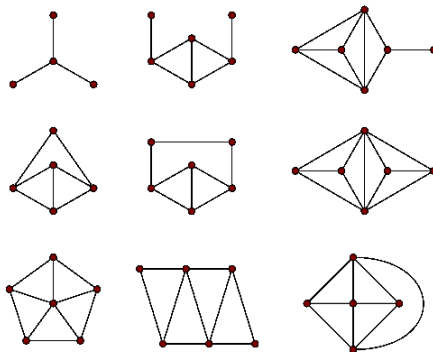
Notice that these classes can be considered as generalizations of interval graphs...

Complexity of recognition

Theorem 9.8. The class *recognition problem* – to decide whether a given abstract graph belongs to the specified class, is

- *polynomial time* solvable for INT, line graphs, CA, and CIR; \square
- and *NP-hard* for DISC, unit-DISC, BOX (in any dimension ≥ 2). \square

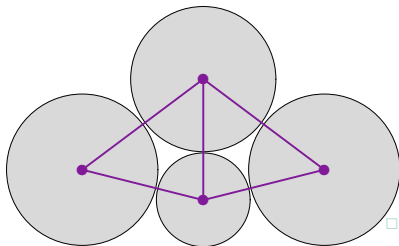
Theorem 9.9. A graph G is a *line graph* of a simple graph if, and only if, G does not contain any of the following induced subgraphs:



Contact (touching) graphs

Considering intersection graphs of geometric objects, it is sometimes natural to define the following restriction:

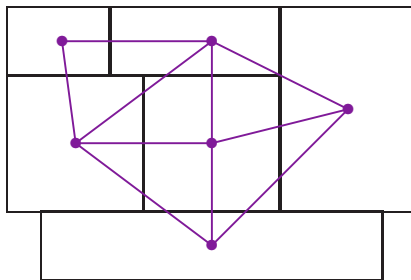
- *Contact graphs* are the graphs having an intersection representation such that the objects “do not overlap” (formally, their topol. interiors are pairwise disjoint).



A particularly beautiful result of Koebe reads:

Theorem 9.10. A graph G is *planar* if, and only if, G is a contact graph of discs in the plane (a *coin graph*).

- *Rectangular duals* – another example: Those are the graphs having a contact representation by a collection of (non-overlap.) *axis-parallel rectangles* in the plane.



- Note; meeting of four rectangles in one point is disallowed! □

Fact: Only *planar* graphs can have rectangular duals, but not all planar graphs do have. □

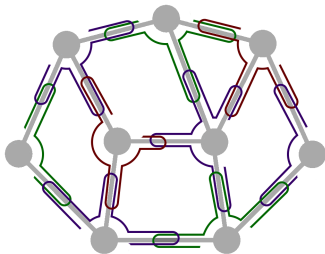
- In a strict sense, rectangular dual repres. must “fill the plane *without holes*”.

Fact: Strict rectangular duals always represent planar *quasi-triangulations* (all faces except the outer one are *triangles*).

9.4 Curve and line segment intersection graphs

- *String graphs* are the intersection graphs of simple curves in the plane.

For example, every planar graph is a string graph, see:



□

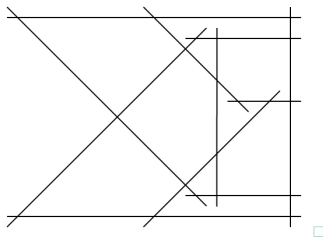
On the other hand, not all graphs are string graphs (the smallest non-string graphs have 12 vertices.) □

Moreover, the structure of string representations can be very complicated.

Proposition 9.11. *There exist string graphs such that every their representation contains a pair of curves having exponentially many intersections.*

- *Segment intersection graphs* are the intersection graphs of straight line segments in the plane.

For example, see:



This is a proper subclass of string graphs, and the structure is again quite complicated. For instance, there exist segment intersection graphs such that every their representation requires double-exponential precision of segment coordinates. □

Theorem 9.12. *The class recognition problems are NP-hard for both string and segment intersection graphs.*

On the other hand, the following two statements are highly nontrivial and their proofs have been searched for many years.

Theorem 9.13. *The recognition problem of string graphs is in NP.* □

Theorem 9.14. *Every planar graph is a segment intersection graph.* □

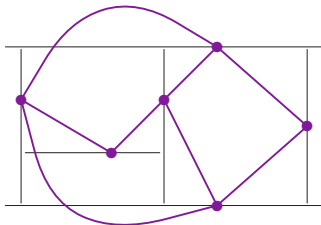
Question. How many “segment slopes” one needs to represent every planar graph as a segment intersection graph?

“Match” graphs

Similarly to coin graphs, one can straightforwardly define a special subclass of segment graphs in which the line segments in an intersection representation are **not** allowed to cross in their interior points (i.e., having **pairwise disjoint interiors**).

- The aforementioned class is called the class of *segment contact graphs*. □

Theorem 9.15. *A graph G is a segment contact graph of only vertical and horizontal segments if, and only if, G is a planar bipartite graph.*



□

Though...

Theorem 9.16. *The recognition problem is **NP-complete** for segment contact graphs.*