$\boxed{\text{št. 11}}$

Př. 1 (5 bodů)

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 3 \end{pmatrix} \longrightarrow \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & -1 & 3-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda) + \underline{1+1} + (1-\lambda) + (1-\lambda) \underline{- (3-\lambda)} =$$

$$= (1-\lambda)^2(3-\lambda) \underbrace{-1+\lambda}_{-(1-\lambda)} + 2(1-\lambda) =$$

$$= (1-\lambda)^2(3-\lambda) + (1-\lambda) = (1-\lambda)\left[(1-\lambda)(3-\lambda)+1\right]$$

$$= (1-\lambda)(3-\lambda-3\lambda+\lambda^2+1) = (1-\lambda)\underbrace{(\lambda^2-4\lambda+4)}_{(\lambda-2)^2} = 0$$

$\Rightarrow \underline{\lambda_{1,2,3} = 1, 2, 2}$

Zkouška: $Tr(A) = 1+1+3 = \lambda_1 + \lambda_2 + \lambda_3$ ✓

$\lambda_1 = 1$:

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \left.\begin{matrix} x_3 = t \\ x_2 = t \\ x_1 = t \end{matrix}\right\} \text{Eigen}(1) = \underline{\text{span}\langle (1,1,1)^T \rangle}$$

$\lambda_{2,3} = 2$:

$$\begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \left.\begin{matrix} x_3 = t \\ x_2 = s \\ x_1 = t-s \end{matrix}\right\} \text{Eigen}(2) = \underline{\text{span}\langle (1,0,1)^T, (-1,1,0)^T \rangle}$$

$\Rightarrow$ Matice A je diagonalizovatelná, protože algebraické násobnosti všech vlastních čísel jsou stejné jako jejich geometrické násobnosti

Př. 2 (5 bodů)

$$V_1 = U_1 = \underline{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$$

$$V_2 = U_2 - \frac{\langle U_2, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1 = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} - \frac{1+3}{1+1+1+1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \underline{\begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}}$$

$$V_3 = U_3 - \frac{\langle U_3, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1 - \frac{\langle U_3, V_2 \rangle}{\langle V_2, V_2 \rangle} V_2 = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} - \frac{2+5+1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{-5-1}{1+4+1}\begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix} = \underline{\begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix}}$$

Zkouška: $\langle V_1, V_2 \rangle = -1+2-1 = 0$ ✓

$\langle V_1, V_3 \rangle = 2-2 = 0$ ✓

$\langle V_2, V_3 \rangle = -2+2 = 0$ ✓

$$\Rightarrow \beta = \underline{\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} \right)}$$

Př. 3 (5 bodů)

$$\underbrace{\begin{pmatrix} M_{t+1} \\ V_{t+1} \end{pmatrix}}_{x_{t+1}} = \begin{array}{c|cc} & M_t & V_t \\ M_{t+1} & & \\ V_{t+1} & & \end{array} \underbrace{\begin{pmatrix} 0{,}85 & 0{,}05 \\ 0{,}15 & 0{,}95 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} M_t \\ V_t \end{pmatrix}}_{x_t}, \quad M_0 + V_0 = 400\,000, \quad \underbrace{\begin{pmatrix} M_0 \\ V_0 \end{pmatrix}}_{x_0}$$

$$A^t = P \cdot D^t \cdot P^{-1}$$

nebo: víme, že u stochastické matice je $\underline{\lambda_1 = 1}$

zároveň víme, že $Tr(A) = \lambda_1 + \lambda_2 = 1 + \lambda_2$

$\Rightarrow Tr(A) = 0{,}85 + 0{,}95 = 1{,}8 = 1 + \lambda_2 \Rightarrow \underline{\underline{\lambda_2 = 0{,}8}}$

$$\begin{vmatrix} 0{,}85-\lambda & 0{,}05 \\ 0{,}15 & 0{,}95-\lambda \end{vmatrix} = (0{,}85-\lambda)(0{,}95-\lambda) - 0{,}05 \cdot 0{,}15 = 0{,}8075 - 0{,}85\lambda - 0{,}95\lambda + \lambda^2 - 0{,}0075 =$$

$$= \lambda^2 - 1{,}8\lambda + 0{,}8 = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{1{,}8 \pm \sqrt{1{,}8^2 - 4 \cdot 0{,}8}}{2} = \frac{1{,}8 \pm \sqrt{0{,}04}}{2} = \frac{1{,}8 \pm 0{,}2}{2} = \left\langle \begin{array}{l} \underline{\underline{1}} \\ \underline{\underline{0{,}8}} \end{array} \right.$$

kontrola: právě jedno vl. číslo $= 1$,

zbytek $|\lambda| < 1$ ✓

$\lambda_1 = 1$:

$$\begin{pmatrix} -0{,}15 & 0{,}05 \\ 0{,}15 & -0{,}05 \end{pmatrix} \sim \begin{pmatrix} -0{,}15 & 0{,}05 \\ 0 & 0 \end{pmatrix} \Rightarrow \left. \begin{array}{l} x_2 = t \\ x_1 = \frac{5}{15} t = \frac{1}{3} t \end{array} \right\} r_1 = (1, 3)^T$$

$\lambda_2 = 0{,}8$:

$$\begin{pmatrix} 0{,}05 & 0{,}05 \\ 0{,}15 & 0{,}15 \end{pmatrix} \Rightarrow \left. \begin{array}{l} x_2 = t \\ x_1 = -t \end{array} \right\} r_2 = (-1, 1)^T$$

$$\Rightarrow A^t = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1^t & 0 \\ 0 & 0{,}8^t \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \cdot \frac{1}{4}; \quad \lim_{t \to \infty} A^t = \underbrace{\frac{1}{4} \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\frac{1}{4} \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$$

$$\lim_{t \to \infty} x_t = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \cdot x_0 = \begin{pmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{pmatrix} \begin{pmatrix} M_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} 1/4\,(M_0 + V_0) \\ 3/4\,(M_0 + V_0) \end{pmatrix} = \underline{\begin{pmatrix} 100\,000 \\ 300\,000 \end{pmatrix}}$$

$\Rightarrow$ Populace se v Brně nakonec ustálí tak, že 100000 obyvatel bude žít ve městě a zbylých 300 000 obyvatel na předměstí st. 12

Př. 1 (5 bodů)

$$A=\begin{pmatrix}0&-1&-1\\1&2&1\\2&1&3\end{pmatrix} \rightarrow \begin{vmatrix}-\lambda&-1&-1\\1&2-\lambda&1\\2&1&3-\lambda\end{vmatrix} = -\lambda(2-\lambda)(3-\lambda)-1-2+2(2-\lambda)+\lambda+(3-\lambda)=$$

$$=-\lambda(2-\lambda)(3-\lambda)\underbrace{-3+\lambda}_{-(3-\lambda)}+\cancel{(3-\lambda)}+2(2-\lambda)=$$

$$=(2-\lambda)[-\lambda(3-\lambda)+2]=(2-\lambda)\underbrace{(-3\lambda+\lambda^2+2)}_{(\lambda-2)(\lambda-1)}=0$$

$\Rightarrow \lambda_{1,2,3}=1,2,2$

Zkouška: $Tr(A)=2+3=\lambda_1+\lambda_2+\lambda_3$ ✓

$\lambda_1=1$

$$\begin{pmatrix}-1&-1&-1\\1&1&1\\2&1&2\end{pmatrix}\sim\begin{pmatrix}1&1&1\\0&-1&0\\0&0&0\end{pmatrix} \rightarrow \left.\begin{matrix}x_3=t\\x_2=0\\x_1=-t\end{matrix}\right\} \mathrm{Eigen}(1)=\mathrm{span}\langle(1,0,-1)^T\rangle$$

$\lambda_{2,3}=2$

$$\begin{pmatrix}-2&-1&-1\\1&0&1\\2&1&1\end{pmatrix}\sim\begin{pmatrix}1&0&1\\0&1&-1\\0&0&0\end{pmatrix} \rightarrow \left.\begin{matrix}x_3=t\\x_2=t\\x_1=-t\end{matrix}\right\} \mathrm{Eigen}(2)=\mathrm{span}\langle(-1,1,1)^T\rangle$$

$\Rightarrow$ Matice A není diagonalizovatelná, protože algebraické násobnosti nejsou stejné jako geometrické (u $\lambda_{2,3}=2$ je $a.n=2$, $g.n=1$)

Př. 2 (5 bodů)

$$V_1=U_1=\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

$$V_2=U_2-\frac{\langle U_2,V_1\rangle}{\langle V_1,V_1\rangle}V_1=\begin{pmatrix}1&0\\3&0\end{pmatrix}-\frac{1+3}{1+1+1+1}\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}1&0\\3&0\end{pmatrix}-\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}0&-1\\2&-1\end{pmatrix}$$

Zkouška: $\langle V_1,V_2\rangle=0-1+2-1=0$ ✓

$$V_3=U_3-\frac{\langle U_3,V_1\rangle}{\langle V_1,V_1\rangle}V_1-\frac{\langle U_3,V_2\rangle}{\langle V_2,V_2\rangle}V_2=\begin{pmatrix}-1&8\\1&0\end{pmatrix}-\frac{-1+8+1}{4}\begin{pmatrix}1&1\\1&1\end{pmatrix}-\frac{-8+2}{1+4+1}\begin{pmatrix}0&-1\\2&-1\end{pmatrix}$$

$$=\begin{pmatrix}-1&8\\1&0\end{pmatrix}-2\begin{pmatrix}1&1\\1&1\end{pmatrix}-\frac{-6}{6}\begin{pmatrix}0&-1\\2&-1\end{pmatrix}=\begin{pmatrix}-3&5\\1&-3\end{pmatrix}$$

Zkouška: $\langle V_2,V_3\rangle=0-5+2+3=0$ ✓

$\langle V_1,V_3\rangle=-3+5+1-3=0$ ✓

$$\Rightarrow \beta=\left(\begin{pmatrix}1&1\\1&1\end{pmatrix},\begin{pmatrix}0&-1\\2&-1\end{pmatrix},\begin{pmatrix}-3&5\\1&-3\end{pmatrix}\right)$$

Př. 3 (5 bodů)

$$\underbrace{\begin{pmatrix} M_{k+1} \\ V_{k+1} \end{pmatrix}}_{x_{k+1}} = \begin{array}{c} \\ M_{k+1} \\ V_{k+1} \end{array}\!\!\begin{array}{c} \begin{array}{cc} M_k & V_k \end{array} \\ \underbrace{\begin{pmatrix} 0{,}9 & 0{,}1 \\ 0{,}1 & 0{,}9 \end{pmatrix}}_{A} \end{array} \cdot \underbrace{\begin{pmatrix} M_k \\ V_k \end{pmatrix}}_{x_k}, \quad M_0 + V_0 = 400\,000, \quad \begin{pmatrix} M_0 \\ V_0 \end{pmatrix} = x_0$$

$$A^k = P \cdot D^k \cdot P^{-1}$$

nebo: víme, že u stochastické matice je $\underline{\lambda_1 = 1}$
zároveň víme, že $Tr(A) = \lambda_1 + \lambda_2 = 1 + \lambda_2$
$\Rightarrow Tr(A) = 0{,}9 + 0{,}9 = 1{,}8 = 1 + \lambda_2 \Rightarrow \underline{\lambda_2 = 0{,}8}$

$$\begin{vmatrix} 0{,}9-\lambda & 0{,}1 \\ 0{,}1 & 0{,}9-\lambda \end{vmatrix} = (0{,}9-\lambda)^2 - 0{,}1^2 = 0{,}81 - 1{,}8\lambda + \lambda^2 - 0{,}01 = \lambda^2 - 1{,}8\lambda + 0{,}8 = 0$$

$$\lambda_{1,2} = \frac{1{,}8 \pm \sqrt{1{,}8^2 - 4 \cdot 0{,}8}}{2} = \frac{1{,}8 \pm \sqrt{3{,}24 - 3{,}2}}{2} = \frac{1{,}8 \pm \sqrt{0{,}04}}{2} =$$

$$= \frac{1{,}8 \pm 0{,}2}{2} = \begin{cases} \underline{1} \\ \underline{\underline{0{,}8}} \end{cases}$$

... kontrola: právě 1 vl. číslo $= 1$ ✓
zbylá vl. čísla $|\lambda| < 1$ ✓

$\lambda_1 = 1$

$$\begin{pmatrix} -0{,}1 & 0{,}1 \\ 0{,}1 & -0{,}1 \end{pmatrix} \sim \begin{pmatrix} -0{,}1 & 0{,}1 \\ 0 & 0 \end{pmatrix} \to \left. \begin{matrix} x_2 = t \\ x_1 = t \end{matrix} \right\} v_1 = (1, 1)^T$$

$\lambda_2 = 0{,}8$

$$\begin{pmatrix} 0{,}1 & 0{,}1 \\ 0{,}1 & 0{,}1 \end{pmatrix} \to \left. \begin{matrix} x_2 = t \\ x_1 = -t \end{matrix} \right\} v_2 = (-1, 1)^T$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\to A^k = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1^k & 0 \\ 0 & 0{,}8^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{2}; \quad \lim_{k \to \infty} A^k = \frac{1}{2} \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}$$

$$\lim_{k \to \infty} x_k = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \frac{1}{2} \cdot x_0 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} M_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} 1/2\,(M_0 + V_0) \\ 1/2\,(M_0 + V_0) \end{pmatrix} = \underline{\underline{\begin{pmatrix} 200\,000 \\ 200\,000 \end{pmatrix}}}$$

$\Rightarrow$ Populace se v Brně nakonec ustálí tak, že 200 000 obyvatel bude žít ve městě a zbylých 200 000 obyvatel na předměstí.