Příklad

Určete
$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

Řešení

Dosazením za $x=0$ dostaneme, že tato limita je typu $\left|\frac{0}{0}\right|$.

rozšíření zlomku $\qquad \sin^2 x+\cos^2 x=1$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\left(\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\right) =$$
$$= \lim_{x\to 0}\left(\frac{1-\cos^2 x}{x}\cdot\frac{1}{1+\cos x}\right) = \lim_{x\to 0}\left(\frac{\sin^2 x}{x}\cdot\frac{1}{1+\cos x}\right) =$$
$$= \lim_{x\to 0}\left(\underbrace{\frac{\sin x}{x}}_{\to 1}\cdot\underbrace{\sin x}_{\to 0}\cdot\underbrace{\frac{1}{1+\cos x}}_{\to\frac{1}{2}}\right) = 1\cdot 0\cdot\frac{1}{2} = 0.$$

10 12-11:53

Př: $\frac{e^x-1}{x} \quad x\to 0$

$$\left(1+\frac{1}{n}\right)^n \nearrow e \swarrow \left(1+\frac{1}{n}\right)^{n+1}$$

$$\Rightarrow \left(1+\frac{1}{n+1}\right)^{n+1} < e < \left(1+\frac{1}{n-1}\right)^n \quad \forall n\in\mathbb{N}\setminus\{1\}$$

$$\frac{1}{n+1} < x \leq \frac{1}{n}$$

$$e^x > e^{\frac{1}{n+1}} > 1+\frac{1}{n+1} \geq 1+\frac{x}{x+1}$$

$$\Updownarrow$$

$$n+1 \leq \frac{x+1}{x} = 1+\frac{1}{x}$$

$$\Updownarrow$$

$$n \leq \frac{1}{x} \Leftrightarrow x \leq \frac{1}{n}$$

$$e^x \geq 1+\frac{x}{x+1}$$

$$\frac{e^x-1}{x} \geq \frac{1}{x+1}$$

10 12-12:04

Příklad

Určete
$a\in\mathbb{R}^+\setminus\{1\}$
$$\lim_{x\to 0}\frac{a^x-1}{x}.$$

$$a^x \xrightarrow{\ln} \ln(a^x) = x\ln a$$
$$\xrightarrow{e^{\cdot\cdot\cdot}} e^{x\cdot\ln a}$$

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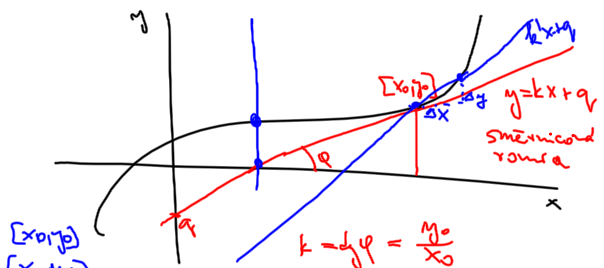

$[x_0,y_0]$
$[x_1,y_1]$

$$k = \operatorname{tg}\varphi = \frac{y_0}{x_0}$$

$$\Delta x = x_1 - x_0$$
$$\Delta y = y_1 - y_0 \Rightarrow k' = \frac{\Delta y}{\Delta x}$$

$$x_1 \to x_0$$
$$\Delta x \to 0$$
$$\Downarrow ?$$
$$k = \frac{\Delta y}{\Delta x} \longrightarrow \text{derivace v bodě } x_0$$
$$\frac{dy}{dx}$$

10 12-12:21

Příklad

Určete rovnici tečny ke grafu funkce $f(x)=1/x$ v bodě $x_0=1$.

rovnice tečny k $f(x)$ v $[x_0,y_0]$ $\quad y_0=f(x_0)$

$$\boxed{y-y_0 = f'(x_0)\cdot(x-x_0)}$$

směrnice tečny

v bodě x_0:

$$\lim_{x\to x_0}\frac{\frac{1}{x}-\frac{1}{x_0}}{x-x_0} = \lim_{x\to x_0}\frac{\frac{x_0-x}{x x_0}}{x-x_0} = \lim_{x\to x_0}\frac{-1}{x x_0} = -\frac{1}{x_0^2}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

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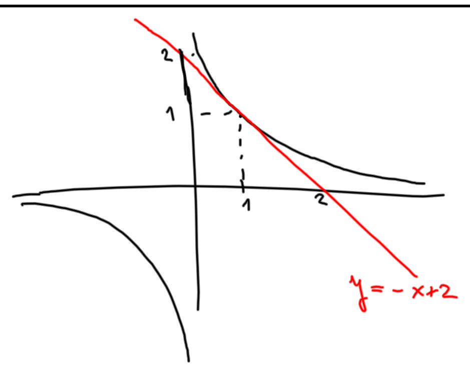

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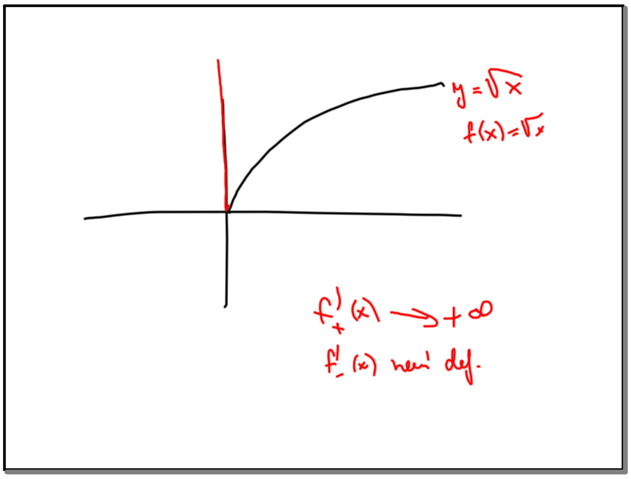

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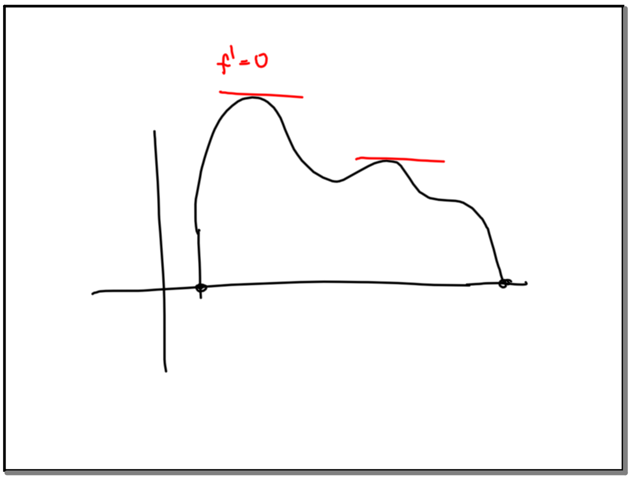
$f(x)=x^n$

$$\lim_{x\to x_0}\frac{x^n-x_0^n}{x-x_0}=\lim_{x\to x_0}\frac{\cancel{(x-x_0)}(x^{n-1}+x^{n-2}x_0+\dots+x\cdot x_0^{n-2}+x_0^{n-1})}{\cancel{x-x_0}}=$$

$$=\underline{\underline{n\cdot x_0^{n-1}}}$$

10 12-12:41



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Př.: $y=\sqrt[3]{x}$

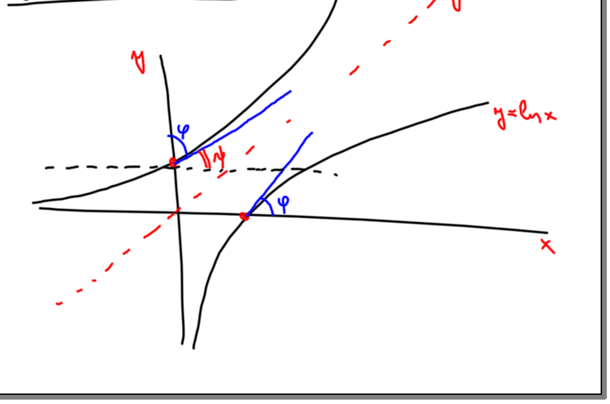
$y^3=x \quad |\,\text{deriv.}$

$$3y^2=\lim_{\Delta y\to 0}\frac{\Delta x}{\Delta y}=\frac{dx}{dy}$$

Chci: $$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{1}{3y^2}=\frac{1}{3(\sqrt[3]{x})^2}=\frac{1}{3}x^{-\frac{2}{3}}$$

10 12-13:07

geometrický pohled :



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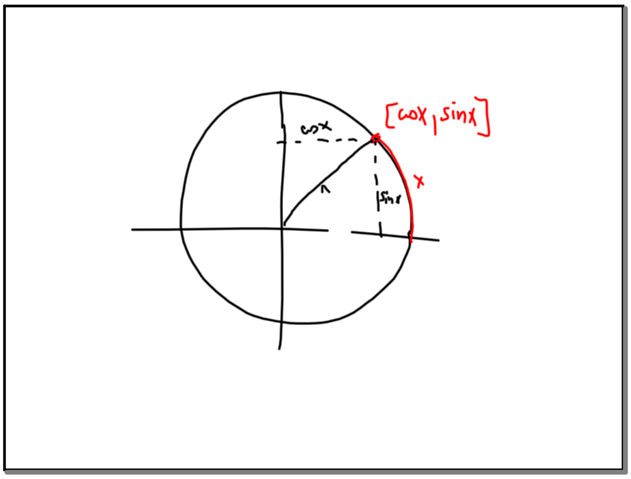

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$y = x^{\frac{1}{q}} = \sqrt[q]{x} \qquad q \in \mathbb{N}$

$y^q = x \quad |$ derivujeme $\qquad r := \frac{1}{q}$

$$q y^{q-1} = \frac{dx}{dy} \overset{\text{V inv.}}{\Longrightarrow} \frac{dy}{dx} = \frac{1}{q \cdot y^{q-1}} =$$

$$= \frac{1}{q \cdot (x^{\frac{1}{q}})^{q-1}} = \frac{1}{q \cdot x^{1-\frac{1}{q}}} =$$

$$= \frac{1}{q} \cdot x^{\frac{1}{q}-1} = r x^{r-1}$$

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$$(\operatorname{tg} x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \underline{\underline{\frac{1}{\cos^2 x}}}$$

$y = \ln x \iff x = e^y$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x} \Leftarrow \frac{dx}{dy} = (e^y)' = e^y$$

$$(\ln x)' = \frac{1}{x}$$

10 12-13:29

$y = \arcsin x \qquad x = \sin y$

$$\frac{dy}{dx} = \frac{1}{\cos y} \Leftarrow \frac{dx}{dy} = \cos y$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{1}{\cos(\arcsin x)}$$

$y = \operatorname{arctg} x \qquad x = \operatorname{tg} y$

$$\frac{dx}{dy} = \frac{1}{\cos^2 y} = \frac{\sin^2 y + \cos^2 y}{\cos^2 y}$$

$$\frac{dy}{dx} = \cos^2 y$$

$$= \dots = \frac{1}{1+x^2}$$

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