DIFERENCIÁLNÍ RCE

např. $y'x + y^2 + x = 0$

$$y' = \frac{dy}{dx}$$

PŘ. 1 $y' = 2x+1$; $y(0)=0$

$$\frac{dy}{dx} = 2x+1 \quad \Big/ \cdot dx$$
$$dy = (2x+1)\,dx \quad \Big/ \int$$
$$\int dy = \int (2x+1)\,dx$$
$$\boxed{y = x^2 + x + c}$$

$y(0)=0$ (spočítáme c)
$$0 = 0^2 + 0 + c$$
$$c = 0$$

Hledaná fce $y = x^2 + x + 0$

je to parabola
$$y = x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4}$$
$$= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$$
$$V = \left[-\frac{1}{2}, -\frac{1}{4}\right]$$

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PŘ. 2

(Řeší se metodou separace proměnných)
$$y' = f(x)\cdot g(y)$$
$$\frac{dy}{dx} = f(x)g(y)$$
$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

$$y'(1-x^2) + y = 0$$
$$y'(1-x^2) = -y$$
$$y' = -\frac{y}{1-x^2} \qquad f(x) = -\frac{1}{1-x^2} \quad g(y) = y$$
$$\frac{dy}{dx} = -\frac{y}{1-x^2}$$
$$\int \frac{dy}{y} = \int\left(-\frac{dx}{1-x^2}\right)$$

$$\int \frac{dx}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} = \int \frac{1}{2(1-x)} + \frac{-1}{2(1+x)}\,dx =$$
$$A + Ax + B - Bx = -1$$
$$A + B = -1$$
$$A - B = 0 \qquad A = B = -\frac{1}{2}$$
$$= +\frac{1}{2}\ln|1-x| - \frac{1}{2}\ln|1+x| + c$$
$$= \frac{1}{2}\ln\left|\frac{1-x}{1+x}\right| + c$$

$$\ln|y| = \frac{1}{2}\ln\left|\frac{1-x}{1+x}\right| + c$$
$$\ln|y| = \ln\sqrt{\left|\frac{1-x}{1+x}\right|} + c$$
$$y = e^{\ln\sqrt{\frac{1-x}{1+x}} + c}$$
$$y = e^{\ln\sqrt{\frac{1-x}{1+x}}}\cdot k \qquad k = e^c$$
$$y = \sqrt{\frac{1-x}{1+x}}\cdot k$$

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PŘ 3. $2y'\sqrt{x} = y$

$$y' = \frac{y}{2\sqrt{x}}$$
$$\frac{dy}{dx} = \frac{y}{2\sqrt{x}}$$
$$\int \frac{dy}{y} = \int \frac{dx}{2\sqrt{x}} = \int \frac{1}{2}\cdot x^{-\frac{1}{2}}\,dx$$
$$\ln|y| = \frac{1}{2}\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = \sqrt{x} + c$$
$$y = e^{\sqrt{x}}\cdot k \;;\; y = 0$$

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PŘ 4: $y' = 2\sqrt{y}\ln x$; $y(e) = 1$

$$\frac{dy}{dx} = 2\sqrt{y}\ln x$$
$$\int \frac{dy}{\sqrt{y}} = \int 2\ln x\,dx$$
$$\int 2\ln x\,dx = \left| \begin{matrix} u = \ln x & u' = \frac{1}{x} \\ v' = 2 & v = 2x \end{matrix} \right| =$$
$$= 2x\ln x - \int 2\,dx = 2x\ln x - 2x + c$$
$$2\sqrt{y} = 2x\ln x - 2x + c$$
$$\sqrt{y} = x\ln x - x + c$$

POČ. POD.: $y(e) = 1$
$$\sqrt{1} = e\cdot\ln e - e + c$$
$$c = 1$$
$$\boxed{\sqrt{y} = x\ln x - x + 1}$$

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6) $y' = x + y - 1$

$$y' = f(ax + by + c)$$

SUBSTITUCE: $z = ax + by + c$

subs. $z = x + y - 1 \quad \Big| \frac{d}{dx}$ (zderivuju podle x)
$$z' = 1 + y'$$
$$y' = z' - 1$$

$$z' - 1 = z$$
$$z' = z + 1$$
$$\frac{dz}{dx} = z + 1$$
$$\int \frac{dz}{z+1} = \int dx$$
$$\ln|z+1| = x + c$$
$$z + 1 = e^x\cdot k$$
$$\underbrace{x + y - 1}_{z} + 1 = e^x\cdot k$$
$$y = e^x\cdot k - x$$

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LINEÁRNÍ DIFERENCIÁLNÍ ROVNICE (1. řádu)

$$y' = a(x)\cdot y + b(x)$$

je-li $b(x) = 0$ .... homogenní.

JAK SE TO ŘEŠÍ?

1) najdu obecné řešení y_0 tj. řešení $y' = a(x)\cdot y$
$$y_0 = c\cdot e^{\int a(x)dx}$$

2) partikulární řešení y_p (metoda variace konstanty)
$$y = c(x)\cdot e^{\int a(x)dx} \; *$$
$$y' = c'(x)\cdot e^{\int a(x)dx} + c(x)\cdot a(x)\cdot e^{\int a(x)dx}$$

Dosadím y, y' do původního zadání
$$\underbrace{c'(x)\cdot e^{\int a(x)dx} + \cancel{c(x)\cdot a(x)\cdot e^{\int a(x)dx}}}_{y'} = \cancel{a(x)\cdot\underbrace{c(x)\cdot e^{\int a(x)dx}}_{y}} + b(x)$$
$$c'(x)\,e^{\int a(x)dx} = b(x)$$
$$c'(x) = \frac{b(x)}{e^{\int a(x)dx}}$$

Zintegruju obě strany podle x a dostávám $c(x)$, to dosadím do * a mám partikulární řešení y_p. celým řešením je $\boxed{y = y_0 + y_p}$

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PŘ. 8 $y' + xy = x$ LDR

$$y' = \underbrace{-xy}_{a(x)} \underbrace{+x}_{b(x)}$$

OB. ŘEŠENÍ: $y' = -x \cdot y$

$$\frac{dy}{dx} = -x \cdot y$$

$$\int \frac{dy}{y} = \int (-x)\,dx$$

$$\ln|y| = -\frac{x^2}{2} + c$$

$$y_0 = e^{-\frac{x^2}{2}} \cdot k$$

VARIACE KONSTANTY:

$$y = e^{-\frac{x^2}{2}} \cdot k(x)$$

$$y' = -x \cdot e^{-\frac{x^2}{2}} \cdot k(x) + k'(x) \cdot e^{-\frac{x^2}{2}}$$

DOSADÍM DO ZADÁNÍ

$$\underbrace{\cancel{-x \cdot e^{-\frac{x^2}{2}} \cdot k(x)} + k'(x) \cdot e^{-\frac{x^2}{2}}}_{y'} + x \cdot \underbrace{\cancel{e^{-\frac{x^2}{2}} \cdot k(x)}}_{y} = x$$

$$k'(x) \cdot e^{-\frac{x^2}{2}} = x$$

$$k'(x) = \frac{x}{e^{-\frac{x^2}{2}}} = x \cdot e^{\frac{x^2}{2}}$$

$$\int k'(x)\,dx = \int x \cdot e^{\frac{x^2}{2}}\,dx$$

$$k(x) = e^{\frac{x^2}{2}}$$

$$y_p = e^{-\frac{x^2}{2}} \cdot \left(e^{\frac{x^2}{2}}\right) = 1$$

$$y = y_0 + y_p = c \cdot e^{-\frac{x^2}{2}} + 1$$

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PŘ. 9 $y' - \frac{y}{(1+x^2)\operatorname{arctg} x} = \frac{\cos x}{\sqrt{\sin x}} \operatorname{arctg} x$

$$y' = \underbrace{\frac{y}{(1+x^2)\operatorname{arctg} x}}_{a(x)} + \underbrace{\frac{\cos x}{\sqrt{\sin x}} \operatorname{arctg} x}_{b(x)}$$

OB. ŘEŠENÍ $y' = \frac{y}{(1+x^2) \cdot \operatorname{arctg} x}$

$$y = C \cdot e^{\int a(x)\,dx}$$

$$\int a(x)\,dx = \int \frac{1}{(1+x^2)\operatorname{arctg} x}\,dx = \int \frac{\frac{1}{1+x^2}}{\operatorname{arctg} x}\,dx = \ln(\operatorname{arctg} x) \qquad \boxed{e^{\ln x} = x}$$

$$y_0 = C \cdot e^{\ln \operatorname{arctg} x} = C \cdot \operatorname{arctg} x$$

Partikulární:

$$y = c(x) \cdot \operatorname{arctg} x$$

$$y' = c'(x) \cdot \operatorname{arctg} x + c(x) \cdot \frac{1}{1+x^2}$$

$$\underbrace{c'(x)\operatorname{arctg} x + c(x) \cdot \frac{1}{1+x^2}}_{y'} - \frac{c(x) \cdot \cancel{\operatorname{arctg} x}}{(1+x^2) \cdot \cancel{\operatorname{arctg} x}} = \frac{\cos x}{\sqrt{\sin x}} \operatorname{arctg} x$$

$$c'(x)\cancel{\operatorname{arctg} x} = \frac{\cos x}{\sqrt{\sin x}} \cancel{\operatorname{arctg} x}$$

$$\int c'(x)\,dx = \int \frac{\cos x}{\sqrt{\sin x}}\,dx$$

$$\int \frac{\cos x}{\sqrt{\sin x}}\,dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos t\,dt \end{array} \right| = \int \frac{dt}{\sqrt{t}} = 2 \cdot \sqrt{t} = 2 \cdot \sqrt{\sin x}$$

$$c(x) = 2 \cdot \sqrt{\sin x}$$

$$y_p = 2 \cdot \sqrt{\sin x} \cdot \operatorname{arctg} x$$

$$y = c \cdot \operatorname{arctg} x + 2\sqrt{\sin x} \cdot \operatorname{arctg} x$$

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