(1) průběh: $f(x) = \ln(\sin x + 1)$ MB102 - zk2

$D(f)$: $\sin(x)+1 > 0 \iff \sin x > -1 \iff x \notin \{-\frac{\pi}{2} + 2k\pi;\ k \in \mathbb{Z}\}$

$D(f) = \{x \in \mathbb{R};\ x \neq -\frac{\pi}{2} + 2k\pi;\ k \in \mathbb{Z}\}$

Pro $x \in D(f)$ je $\sin x + 1 \in (0, 2\rangle \Rightarrow \ln(\sin x + 1) \in (-\infty, \ln(2)\rangle$

$H(f) = (-\infty, \ln(2)\rangle$

Body nespojitosti jsou právě $\mathbb{R} - D(f)$ (v ostatních bodech je $\sin x$, a tedy i $\ln(\sin x + 1)$ spojitá)

f není sudá, ani lichá

f je periodická s periodou 2π (stačí se proto při vyšetřování průběhu omezit např. na $(-\frac{\pi}{2}, \frac{3\pi}{2})$)

$\boxed{x \in (-\frac{\pi}{2}, \frac{3\pi}{2})}$

znaménko: $\ln(\sin x + 1) > 0 \iff \sin x + 1 > 1 \iff x \in (0, \pi)$

$= 0 \iff$ $x = 0 \vee x = \pi$

derivace: $f'(x) = \frac{\cos x}{\sin x + 1}$; $f'(x) = 0 \iff \cos x = 0 \iff x = \frac{\pi}{2}$

$f'(x) > 0 \iff (\sin x + 1 > 0) \Rightarrow \cos x > 0 \iff x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

f je rostoucí na $(-\frac{\pi}{2}, \frac{\pi}{2})$, klesající na $(\frac{\pi}{2}, \frac{3\pi}{2})$ v $x_0 = \frac{\pi}{2}$ je lokální (i globální) maximum.

2. derivace: $f''(x) = \left(\frac{\cos x}{\sin x + 1}\right)' = \frac{-\sin^2 x - \sin x - \cos^2 x}{(\sin x + 1)^2} = \frac{-1 - \sin x}{(1 + \sin x)^2} = -\frac{1}{1 + \sin x} < 0$

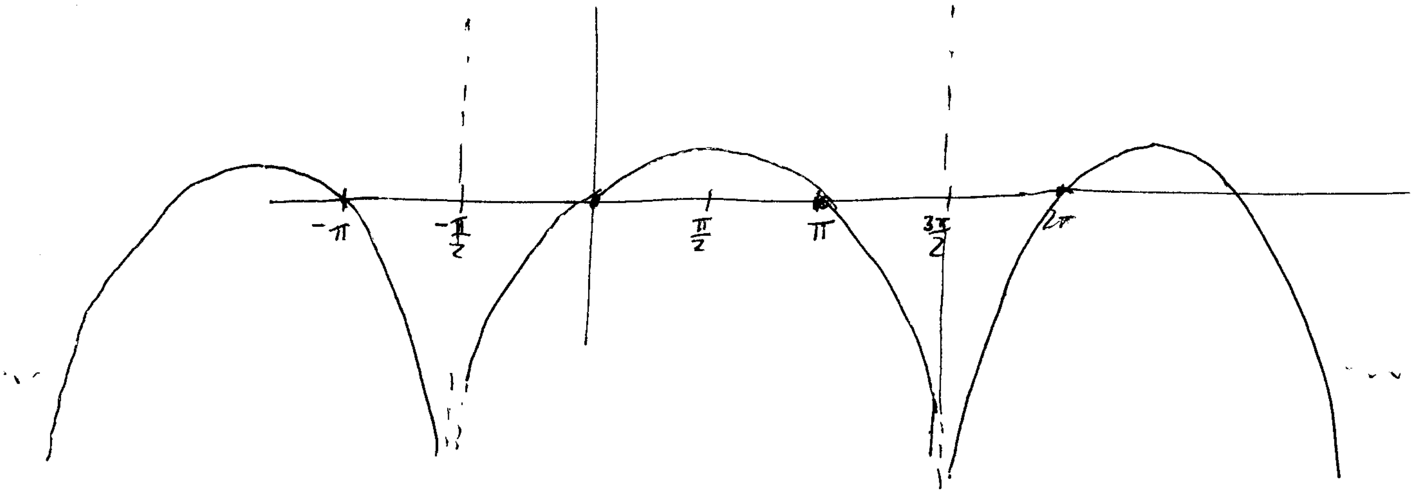
f je tedy konkávní na $(-\frac{\pi}{2}, \frac{3\pi}{2})$, nemá inflexní bod.

asymptoty bez směrnice:

$\lim_{x \to -\frac{\pi}{2}^+} \ln(1 + \sin x) = -\infty$, $\lim_{x \to \frac{3\pi}{2}^-} \ln(1 + \sin x) = -\infty$

$\Rightarrow$ v bodech $\mathbb{R} - D(f)$ jsou asymptoty bez směrnice.

asymptoty se směrnicí nejsou (f není def. na žádném intervalu $(a, +\infty)$ nebo $(-\infty, a)$).



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a) Lagrangeova interpolace $\overset{x_0\ y_0}{[-1,-1]}, \overset{x_1\ y_1}{[3,19]}, \overset{x_2\ y_2}{[-4,5]}$

$$l_0(x) = \frac{(x-3)(x+4)}{(-1-3)(-1+4)}$$

$$l_1(x) = \frac{(x+1)(x+4)}{(3+1)(3+4)}$$

$$l_2(x) = \frac{(x+1)(x-3)}{(-4+1)(-4-3)}$$

$$f(x) = L_2(x) = (-1)l_0(x) + 19 l_1(x) + 5 l_2(x) = +\frac{1}{12}(x^2+x-12) + \frac{19}{28}(x^2+5x+4) + \frac{5}{21}(x^2-2x-3) =$$

$$= x^2 + 3x + 1$$

Lze též řešit soustavou 3 rovnic o 3 neznámých $f(x) = ax^2+bx+c$, $f(-1) = -1$, atd.

b)
$$\int_{-1}^{\sqrt{3}} \frac{x^2+3x+1}{x^2+3}\,dx = \int_{-1}^{\sqrt{3}} dx + \underbrace{\int_{-1}^{\sqrt{3}} \frac{3x-2}{x^2+3}\,dx}_{I_1} = \sqrt{3}+1 + \frac{3}{2}\ln\frac{3}{2} - \frac{2\sqrt{3}}{3}\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$$

$$(x^2+3x+1) : (x^2+3) = 1$$
$$\underline{-(x^2+3)}$$
$$\underline{3x-2}$$

$$I_1 = \frac{3}{2}\int_{-1}^{\sqrt{3}} \frac{2x}{x^2+3}\,dx - 2\int_{-1}^{\sqrt{3}} \frac{dx}{x^2+3} = \frac{3}{2}\Big[\ln(x^2+3)\Big]_{-1}^{\sqrt{3}} - 2\cdot\frac{1}{3}\int_{-1}^{\sqrt{3}} \frac{dx}{\left(\frac{x}{\sqrt{3}}\right)^2+1} =$$

$$= \frac{3}{2}(\ln 6 - \ln 4) - \frac{2}{3}\Big[\sqrt{3}\cdot \operatorname{arctg}\frac{x}{\sqrt{3}}\Big]_{-1}^{\sqrt{3}} = \underline{\underline{\frac{3}{2}\ln\frac{3}{2} - \frac{2}{3}\left(\sqrt{3}\cdot\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\right)}}$$

$$\operatorname{arctg}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

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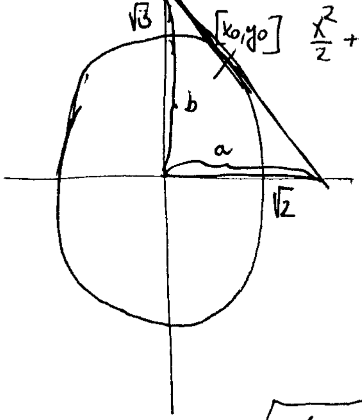

$\frac{x^2}{2}+\frac{y^2}{3}=1$

Stačí uvažovat 1. kvadrant, kde $y=\sqrt{3(1-\frac{x^2}{2})}$

$$y' = \frac{-3x}{2\sqrt{3(1-\frac{x^2}{2})}} = \frac{-\sqrt3\, x}{\sqrt{4-2x^2}}$$

Rovnice tečny jdoucí $[x_0,y_0]$: $y-y_0 = \frac{-\sqrt3\, x_0}{\sqrt{4-2x_0^2}}(x-x_0)$; $\frac{x_0^2}{2}+\frac{y_0^2}{3}=1$

Průsečík s osou x (tj. $y=0$):

$$-y_0 = \frac{-\sqrt3\, x_0}{\sqrt{4-2x_0^2}}(x-x_0)$$

$$\cancel{y_0^2 = \frac{3x^2}{4-2x^2}(x-x_0)^2}$$

$$-\sqrt{3(1-\tfrac{x_0^2}{2})} = \frac{-\sqrt3\, x_0}{\sqrt{4-2x_0^2}}(x-x_0)$$

$$x = x_0 + \frac{2(1-\frac{x_0^2}{2})}{x_0} = a;\quad a = x_0+\frac{2}{x_0}-x_0 = \frac{2}{x_0}$$

Průsečík tečny s osou y (tj. $x=0$):

$$y = y_0 + \frac{-\sqrt3\, x_0}{2\sqrt{1-\frac{x_0^2}{2}}}(-x_0) = y_0 + \frac{\sqrt3\, x_0^2}{2\sqrt{1-\frac{x_0^2}{2}}}$$

$$y = \sqrt3\sqrt{1-\tfrac{x_0^2}{2}} + \frac{\sqrt3\, x_0^2}{2\sqrt{1-\frac{x_0^2}{2}}} = \frac{2\sqrt3(1-\frac{x_0^2}{2})+\sqrt3\, x_0^2}{2\sqrt{1-\frac{x_0^2}{2}}} =$$

$$= \frac{2\sqrt3}{2\sqrt{1-\frac{x_0^2}{2}}} = \frac{\sqrt3}{\sqrt{1-\frac{x_0^2}{2}}} = b$$

Obsah Δ: $S = \frac{ab}{2} = \frac12\cdot\frac{2}{x_0}\cdot\frac{\sqrt3}{\sqrt{1-\frac{x_0^2}{2}}} = \frac{\sqrt3}{x_0\sqrt{1-\frac{x_0^2}{2}}} = \sqrt3\cdot\frac{1}{x_0}\cdot\frac{1}{\sqrt{1-\frac{x_0^2}{2}}} = \sqrt3\cdot\frac{1}{x_0}\cdot(1-\frac{x_0^2}{2})^{-\frac12}$

$S'=0$; $\sqrt3\left(-\frac{1}{x_0^2}\cdot\frac{1}{\sqrt{1-\frac{x_0^2}{2}}} + \frac{1}{x_0}\cdot(-\frac12)\cdot(1-\frac{x_0^2}{2})^{-\frac32}\cdot(-x_0)\right) = 0$

$$+\frac{1}{x_0^2\sqrt{1-\frac{x_0^2}{2}}} = \frac{1}{2\sqrt{(1-\frac{x_0^2}{2})^3}} \iff 2(1-\tfrac{x_0^2}{2}) = x_0^2$$

$$x_0^2 = 1 \iff x_0 = 1,\ y_0=\sqrt{\tfrac32}$$

(4) Taylor

$f(x) = \cos x$

$g(x) = e^{-\frac{x^2}{2}}$

$$T^6_{f,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!}$$

$$T^6_{g,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$$

$$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4} = \lim_{x \to 0} \frac{T^6_{f,0}(x) - T^6_{g,0}(x)}{x^4} = \lim_{x \to 0} \frac{\frac{x^4}{24} - \frac{x^6}{6!} - \frac{x^4}{8} + \frac{x^6}{48}}{x^4} =$$

$$= \frac{1}{24} - \frac{1}{8} = -\frac{1}{12}.$$