① Průběh funkce $f(x) = \operatorname{arctg} \frac{x-1}{x}$

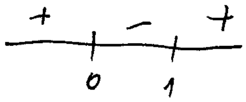
pis-Z83 M310
2011 / 2012

$D(f) = \mathbb{R} \setminus \{0\}$

spojitá na $D(f)$, $\lim_{x \to 0^-} f(x) = \frac{\pi}{2}$, $\lim_{x \to 0^+} f(x) = -\frac{\pi}{2}$; skoková nespojitost

není sudá ani lichá, není periodická

$f(x) > 0 \iff \frac{x-1}{x} > 0 \iff x < 0 \vee x > 1$



$f(x) = 0 \iff \underline{x = 1}$

$$f'(x) = \frac{1}{1+\left(\frac{x-1}{x}\right)^2} \cdot \left(1 - \frac{1}{x}\right)' = \frac{1}{x^2\left(1+\frac{(x-1)^2}{x^2}\right)} = \frac{1}{(x-1)^2 + x^2} = \frac{1}{2x^2 - 2x + 1}$$

$f'(x) > 0 \quad \forall x \in D(f)$

$$f''(x) = \left[(2x^2 - 2x + 1)^{-1}\right]' = -\frac{4x-2}{(2x^2-2x+1)^2}$$

$f''(x) > 0 \iff 4x - 2 < 0 \iff x < \frac{1}{2}$ (konvexní na $(-\infty, 0) \cup (0, \frac{1}{2})$; konkávní $(\frac{1}{2}, \infty)$)

inflexní bod $\underline{x = \frac{1}{2}}$ $\quad f(\frac{1}{2}) = \operatorname{arctg} -1 = -\frac{\pi}{4}$

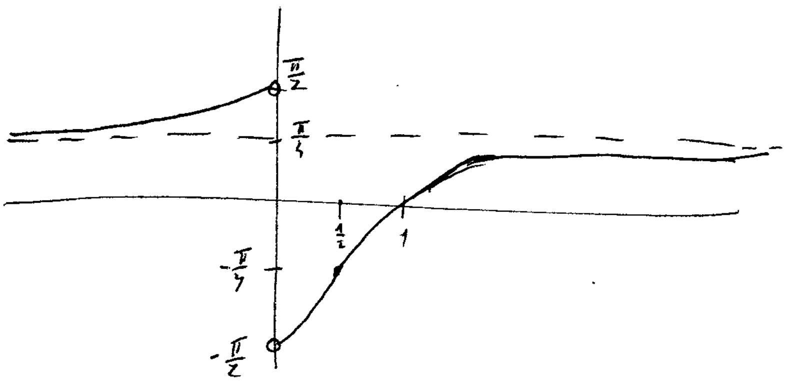
asymptota bez směrnice není (kandidát $x = 0$ nevyhovuje)

asymptota se směrnicí:

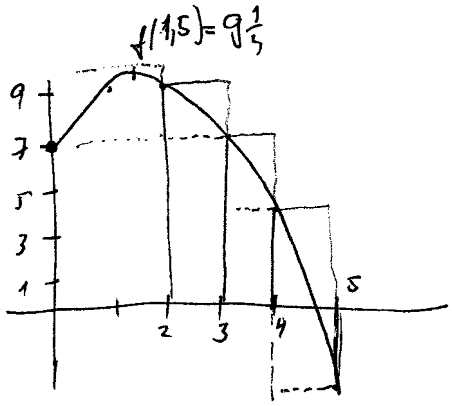
$$\lim_{x \to \pm\infty} \frac{\operatorname{arctg} \frac{x-1}{x}}{x} = \left| \frac{\text{omez.}}{\pm\infty} \right| = 0$$

$$\lim_{x \to \pm\infty} \operatorname{arctg} \frac{x-1}{x} = \operatorname{arctg} 1 = \frac{\pi}{4}.$$

as. s. sm. je $\underline{y = \frac{\pi}{4}}$



(2) $f(x) = -x^2+3x+7$



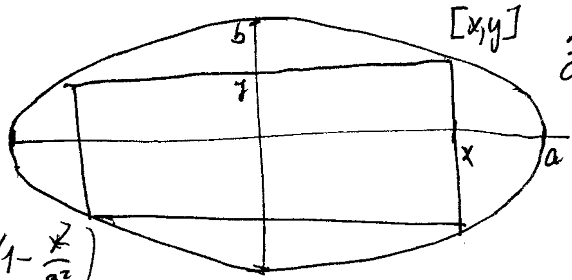
a) $s(D,f) = 7\cdot 2+7\cdot 1+3\cdot 1+(-3)\cdot 1 = 21$

2b $S(D,f) = 9\frac{1}{4}\cdot 2+9\cdot 1+7\cdot 1+3\cdot 1 = 37{,}5$

b) $\int_0^5 f(x)\,dx = 30\frac{5}{6}$

2b

(3)



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$

Lze předp. $x,y>0$. Maximalizujeme obsah $\frac{S}{4} = x\cdot y$, tj. $f(x) = xb\sqrt{1-\frac{x^2}{a^2}}$ ... 2

$f'(x) = b\sqrt{1-\frac{x^2}{a^2}} - b\frac{x^2}{a^2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}$

$f'(x)\geq 0 \iff 1-\frac{x^2}{a^2} \geq \frac{x^2}{a^2} \iff 2x^2 \leq a^2 \overset{x>0}{\iff} \underline{\underline{x \leq \frac{\sqrt{2}}{2}a}}$

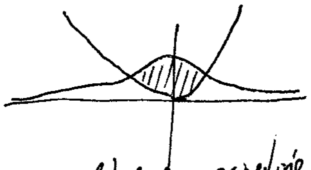
Maximum nastává pro $x_0 = \frac{\sqrt{2}}{2}a$ ($f'(x)>0$ pro $x<x_0$, $f'(x)<0$ pro $x>x_0$)

$y_0 = \frac{\sqrt{2}}{2}b$.

$S_{max} = 4x_0y_0 = \underline{\underline{2ab}}$

(4) a) $f(x) = \frac{2}{1+x^2}$

$g(x) = x^2$



sudé funkce, symetrie podle y

$S = 2\int_0^1\left(\frac{2}{1+x^2} - x^2\right)dx =$

$= 2\left[2\operatorname{arctg} x - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{\pi}{2}-\frac{1}{3}\right) = \underline{\underline{\pi - \frac{2}{3}}}$ 2

$\frac{2}{1+x^2} = x^2$

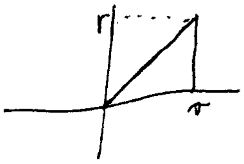
$2 = x^2+x^4$

$x^4+x^2-2=0 \iff x^2 \in \{1, \cancel{-2}\}$

$x^2 = 1 \iff x = \pm 1$.

b) kužel výška v, popř. rotace

$y = \frac{r}{v}\cdot x$ podle x



$V = \pi\int_0^v\left(\frac{r}{v}\cdot x\right)^2 dx = \frac{\pi r^2}{v^2}\cdot\frac{v^3}{3} = \underline{\underline{\frac{\pi r^2 v}{3}}}$ 2

2

$S = 2\pi\int_0^v f(x)\sqrt{1+f'^2(x)}\,dx = 2\pi\int_0^v \frac{r}{v}x\sqrt{1+\frac{r^2}{v^2}}\,dx = 2\pi\cdot\frac{r}{v^2}\sqrt{r^2+v^2}\int_0^v x\,dx = \pi r\sqrt{r^2+v^2}$.