$$y'' - y' = x$$

Char. polynom

$$\lambda^2 - \lambda = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 1$$

obecné řešení homog. rce je $y = c_1 + c_2 e^x$

Teď hledejme řešení nehom. rce:

$y_p = ax^2 + bx$

$$2a - (2ax + b) = x \Rightarrow x^1: -2a = 1 \Rightarrow a = -\frac{1}{2}$$
$$x^0: 2a - b = 0 \Rightarrow b = 2a = -1$$

$$y_p = -\frac{1}{2}x^2 - x$$

$\Rightarrow$ obecné řešení je $y = -\frac{1}{2}x^2 - x + c_1 + c_2 e^x$

$$= \text{99} - \frac{1}{2}x^2 - x + c_1 + c_2 e^x$$

$c_1, c_2 \in \mathbb{R}$

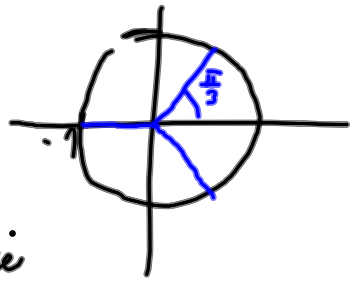
$$y^{(3)} + y = \sin(x)$$

Charak. polynom: $x^3 + 1 = 0$

$x_1 = -1$

$x_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$x_3 = \frac{1}{2} - \frac{\sqrt{3}}{2} i$



⇒ obecné řešení homogenní rce je

$$y = c_1 e^{-x} + c_2 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right)$$

Hledáme partikulární řešení tvaru

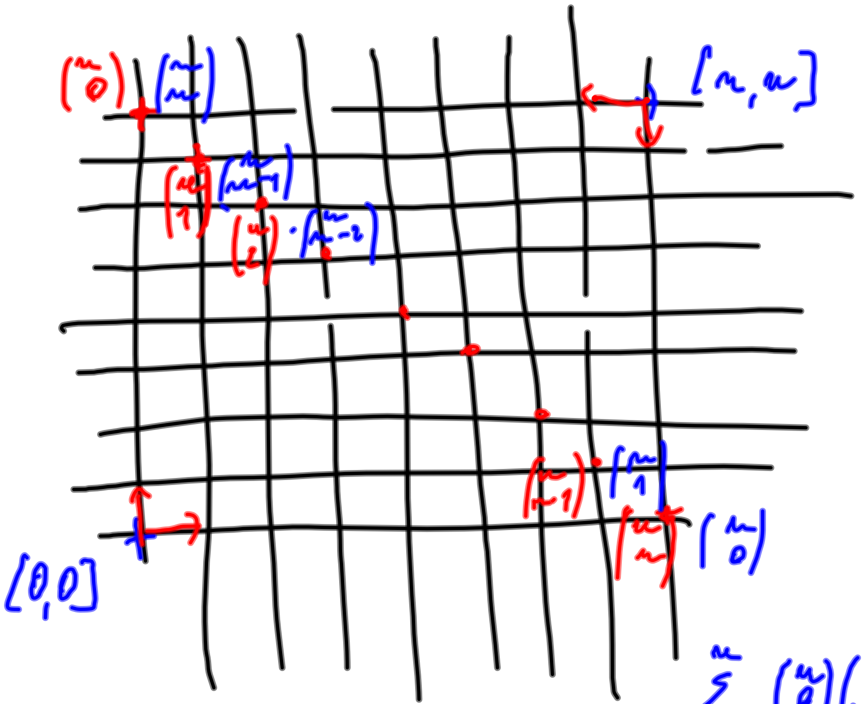
$y_p = a\sin x + b\cos x$:

$y_p^{(3)} = -a\cos x + b\sin x$

$$\boxed{y = \frac{1}{2}\sin x + \frac{1}{2}\cos x + c_1 e^{-x} + c_2 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right)}$$

$$-a\cos x + b\sin x + a\sin x + b\cos x = \sin x$$

$$\left.\begin{array}{l} a + b = 1 \\ a - b = 0 \end{array}\right\} \Rightarrow a = b = \frac{1}{2}$$



Hledaná psť je $\dfrac{\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}}{2^{2n}}$

Určeme pěkně explicitně $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$.

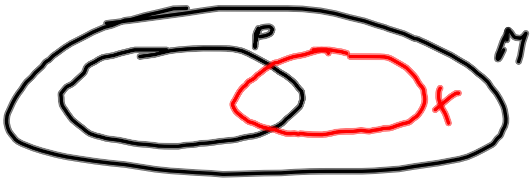
Uvažme polynom $(1+x)^{2n}$. Jeho koeficient u mocniny x^n je $\binom{2n}{n}$, ale také

$$(1+x)^{2n} = (1+x)^n \cdot (1+x)^n =$$

$$= \left[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right] \cdot \left[\binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{n}x^n\right]$$

tedy koeficient u n-té mocniny x je

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$



$|M| = 2n$

$|P| = n$

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n-1}\binom{n}{1} + 1$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

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Př. Určete lokální extrémy funkce

$$f(x,y) = x^3 + y^3 + 3xy \quad \text{na } \mathbb{R}^2$$

$$\frac{\partial f(x,y)}{\partial x} = f_x(x,y) = 3x^2 + 3y = 0 \Rightarrow y = -x^2$$

$$f_y(x,y) = 3y^2 + 3x = 0 \Rightarrow x^4 + x = 0$$

$$x(x^3+1) = 0$$

$$\Rightarrow x_1 = 0,\ x_2 = -1$$

$$\Rightarrow [0,0],\ [-1,-1]$$

$$H_f(x,y) = \begin{pmatrix} 6x & 3 \\ 3 & 6y \end{pmatrix}$$

$$H_f(0,0) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \Rightarrow |H_f(0,0)| < 0$$

$$H_f(-1,-1) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix} : \quad -6<0 \ \& \ 27>0$$

$\Rightarrow$ negativně definitní $\Rightarrow$

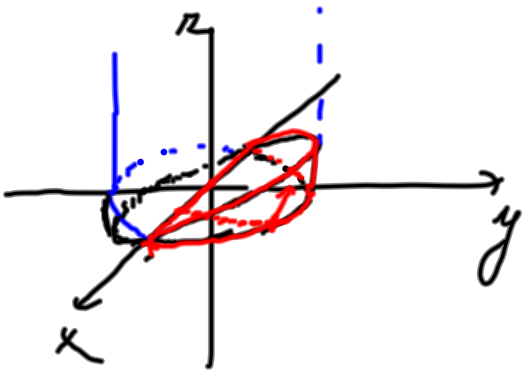
v bodě $[-1,-1]$ je lokální maximum.

Nalezněte extrémy $f(x)$ na elipse $2x^2+y^2=1$ $\quad f(x) = x^3+y^3+3xy$

normála: $(4x, 2y)$

$$\begin{aligned} 3x^2+3y &= \lambda\, 4x \\ 3x+3y^2 &= \lambda\, 2y \\ 2x^2+y^2 &= 1 \end{aligned}$$

Určete objem části prostoru ohraničené rovinami $z=0$ a $z=2y$ uvnitř elipt. válce $1 = 4x^2+y^2$



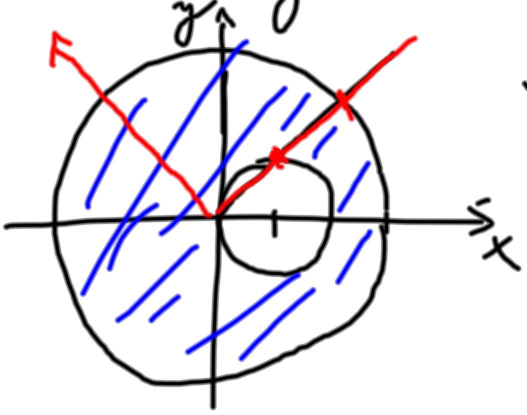
$$V = 4\int_0^{\frac{1}{2}}\int_0^{\sqrt{1-4x^2}} 2y \, dy \, dx =$$

$$= 4\int_0^{\frac{1}{2}} (1-4x^2)\, dx =$$

$$= 4\left[x - \frac{4}{3}x^3\right]_0^{\frac{1}{2}} =$$

$$= 4\cdot\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{4}{3}$$

Určete obsah a těžiště útvaru ležícího mezi kružnicemi $(x-1)^2+y^2=1$ a $x^2+y^2=9$



$$S = 9\pi - \pi = 8\pi$$

$x = r\cos\varphi$, $y = r\sin\varphi$	$\lvert J \rvert = r$
$x^2 - 2x + y^2 = 0$; $r^2 = 2r\cos\varphi$; $r = 2\cos\varphi$	$r = 3$

$$x_T = \frac{1}{S}\iint_{\Omega} x\,dx\,dy =$$

$$= \frac{1}{8\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_a^b r^2\cos\varphi\,dr\,d\varphi =$$

$$= \frac{1}{8\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{2\cos\varphi}^{3} r^2\cos\varphi\,dr\,d\varphi + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^3 r^2\cos\varphi\,dr\,d\varphi\right] =$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{2\cos\varphi}^{3} r^2\cos\varphi \, dr\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(9\cos\varphi - \frac{8}{3}\cos^4\varphi\right) d\varphi =$$

$$= \left[9\sin\varphi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3}\cos^4\varphi =$$

$$\int \cos^2 x - \sin^2 x = \cos 2x$$

$$= 18 - \left[\frac{1}{2}\varphi + \frac{\sin 2\varphi}{4} - \frac{1}{8}\varphi + \frac{\sin 4\varphi}{32}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 18 = -\frac{3}{8}\pi$$

$$\int \cos^4\varphi \, d\varphi = \int \cos^2\varphi\left(1 - \sin^2\varphi\right) d\varphi = \int \cos^2\varphi - \frac{1}{4}\int \sin^2 2\varphi \, d\varphi =$$

$$= \int \frac{1+\cos 2\varphi}{2}\, d\varphi - \frac{1}{4}\int \frac{1-\cos 4\varphi}{2}\, d\varphi =$$

$$= \frac{1}{2}\varphi + \frac{\sin 2\varphi}{4} - \frac{1}{8}\varphi + \frac{\sin 4\varphi}{32}$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{0}^{3} r^2\cos\varphi \, dr\, d\varphi = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 9\cos\varphi \, d\varphi = \left[9\sin\varphi\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -18$$