Druhé cvič. – III – 6.

Numerické:
1) integrace
2) derivace
3) řešení ODE (PDE)

$Tf \equiv \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) \cdot h^n$

řada konverguje x (v pro h)

Zákl. nástroj: Taylorova řada

$f(x+h) = f(x) + f'(x) \cdot h + \frac{1}{2!} f''(x) \cdot h^2 + \frac{1}{3!} f'''(x) h^3 + \dots$

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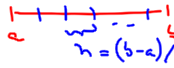
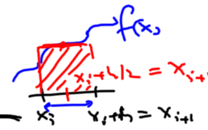
1) Integrace: $I = \int_a^b f(x)\,dx$

$x \in [a,b]$

N dělení



$h = (b-a)/N$



$$\int_{-h/2}^{h/2} f(x_{i+1/2} + y)\,dy = \int_{-h/2}^{h/2} \sum_{n=0}^{\infty} f^{(n)}(x_{i+1/2}) \frac{y^n}{n!}\,dy$$

(potřeb. konv. absolutní a stejnoměrná)

$$= \sum_{n=0}^{\infty} f^{(n)}(x_{i+1/2}) \underbrace{\int_{-h/2}^{h/2} \frac{y^n}{n!}\,dy}_{\frac{1}{(n+1)!}\left[y^{n+1}\right]_{-h/2}^{h/2}} = \sum_{n \text{ sudé}}^{\infty} f^{(n)}(x_{i+1/2}) \frac{2}{(n+1)!} \left(\frac{h}{2}\right)^{n+1}$$

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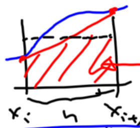
a) lichoběžníkové pravidlo



$A_i = \frac{1}{2}\left(f(x_i) + f(x_{i+1})\right) \cdot h$

$f(x_i) = f(x_i + 1/2 \pm 1/2)$

$$\boxed{\tilde{I} = \sum_{i=0}^{N-1} A_i = \frac{h}{2}\left(f(x_0) + 2 f(x_1) + \dots + 2 f(x_{N-1}) + f(x_N)\right)}$$

$f(x_i + 1/2 \pm 1/2) = f(x_{i+1/2}) \pm \cancel{\frac{h}{2} f'(x_{i+1/2})} + \frac{1}{2!}\left(\frac{h}{2}\right)^2 f''(x_{i+1/2}) \pm \cancel{\dots}$

$A_i = \frac{1}{2}\left(f(x_i) + f(x_{i+1})\right) h = h\left[f(x_{i+1/2}) + \frac{1}{2!}\left(\frac{h}{2}\right)^2 f''(x_{i+1/2}) + O(h^4)\right]$

$\int_{x_i}^{x_{i+1}} f(x)\,dx = h f(x_{i+1/2}) + \frac{2}{3!}\left(\frac{h}{2}\right)^3 f''(x_{i+1/2}) + O(h^5)$

$\int_{x_i}^{x_{i+1}} f(x)\,dx - A_i = \left(-\frac{1}{8} + \frac{1}{24}\right) h^3 f''(x_{i+1/2}) + O(h^5)$

$\boxed{I - \tilde{I}} \simeq \frac{1}{12} N h^3 \|f''\| = \boxed{\frac{1}{12}(b-a) h^2 \|f''\|}$

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b) Simpsonovo pravidlo:

$f_i := f(x_i)$

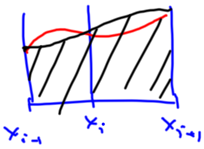
$f_{i+1/2} = f(x_i + h/2)$

$\boxed{f(x_i + y) = f_i + a y + b y^2}$

$\to y = \pm h$: $\left.\begin{array}{l} f_{i+1} = f_i + ah + bh^2 \\ f_{i-1} = f_i - ah + bh^2 \end{array}\right\} \Rightarrow \boxed{bh^2 = \frac{1}{2}(f_{i+1} + f_{i-1} - 2f_i)}$

$A_i = \int_{-h}^{h} (f_i + ay + by^2)\,dy = 2h f_i + \frac{2}{3} b h^3$

$= \frac{1}{3} h (4 f_i + f_{i+1} + f_{i-1})$



$\leadsto$ $I - \tilde{I} \approx \frac{(b-a)}{180} h^4 \|f^{(4)}\|$.

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2) diference:

$f_{i\pm1} = f_i \pm h f_i' + \frac{h^2}{2!} f_i'' \pm \frac{h^3}{3!} f_i''' + \dots$

aproximace f_i' ?

$f_i' = \left(\frac{df}{dx}\right)(x_i) = \frac{f_{i+1} - f_{i-1}}{2h} - \frac{h^2}{3!} f_i''' - \dots$

$f_i' := \frac{1}{2h}(f_{i+1} - f_{i-1})$

alternativní:

$f_i' := \frac{1}{h}(f_{i+1} - f_i)$ ← forward

$f_i' := \frac{1}{h}(f_i - f_{i-1})$ ← backward

aproximace f_i'' ?

$$\boxed{f_i'' \equiv \left(\frac{d^2 f}{dx^2}\right)(x_i) \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}}$$

chyba $\approx \left(\frac{h^2}{12}\right) f_i^{(4)}$

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3) ODE metody: $\frac{dy}{dx} = f(x,y)$

$\boxed{y(x_{i+1})} = y_{i+1} = y_i + h y_i' + O(h^2)$

$= \boxed{y_i + h f(x_i, y_i)}$

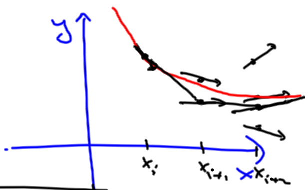
⇒ Eulerova metoda

R diferenciální rovnice 1. řádu

v každém kroku je chyba $\sim O(h^2)$

na intervalu (a,b) máme $N \cong \frac{(b-a)}{h}$ kroků

⇒ $\boxed{\text{chyba} \approx O(h)}$



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$$\boxed{y' = -y \qquad y(0) = 1}$$

$y = Ce^{-x}$

$C = 1$

$\Rightarrow \underline{y = e^{-x}}$

$$\Rightarrow y_{i+1} = y_i - h y_i = (1-h) y_i$$

<u>Milneho metoda</u> — centrální diference

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$$\boxed{y' = f(x,y)}$$

$$y(x+h) = y(x) + h\, y'(x) + \frac{h^2}{2!} y''(x) + \dots$$

$$y_{i+1} = y_i + h \underbrace{\boxed{y_i^{(1)}}}_{= f(x_i, y_i)} + \frac{h^2}{2!} y_i^{(2)} + \frac{h^3}{3!} y_i^{(3)} + \dots$$

$$\Rightarrow y_i^{(2)} = \frac{\partial f}{\partial x}(x_i, y_i) + \frac{\partial f}{\partial y}(x_i, y_i) \cdot f(x_i, y_i)$$

<u>Další metody</u>: prediktor × korektor

<u>Runge – Kutta</u>

$$y_{i+1} = y_i + \alpha_1 h f_i + \alpha_2 h f(x_i + \beta_1 h, y_i + \beta_2 h f_i)$$

$$\leadsto \alpha_2 = 1 - \alpha_1 \quad \beta_1 = \beta_2 = \frac{1}{2(1-\alpha_1)}$$

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