11 Advanced Drawings of Graphs

Since not all graphs are planar, it is natural to ask what can be done about "nice drawing" of the nonplanar ones. Two possible approaches come in mind first:

Either we stick with the definition of a proper *embedding*, and generalize a drawing from the plane to so-called higher surfaces, or
we on the other hand allow adges in a drawing to *cross* each other (but not too often).

We briefly outline these directions now... □

Brief outline of this lecture

- Higher surfaces, and graph embeddings on them.
- Properties of embeddable graphs on surfaces.
- Graph crossing number, crossing minimization.
- Curiosity – the planar cover and planar emulator problems.

11.1 What is a Higher Surface

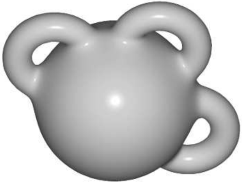
We start with one result of classical topology – the surface classification theorem.

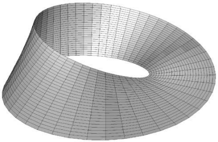
Theorem 11.1. *Every surface (a compact 2-manifold without boundary) is homeomorphic to one of* □

– $\mathcal{S}_0$ *the sphere,*
– $\mathcal{S}_h$ *the sphere with h added handles,*
– $\mathcal{N}_k$ *the sphere with k added crosscaps.*

The operation of adding handles is easily visualized as on the left picture.

Since crosscaps are harder to imagine, we may (almost) equivalently visualize that as adding *Möbius bands*, see on the right.





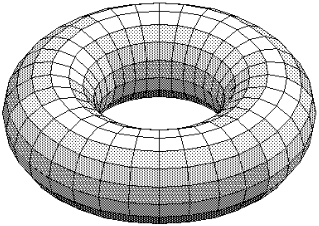
□

Definition: A *crosscap* is a circle (on the surface) such that all its pairs of opposite points are indentified, and the interior of this circuit is removed.

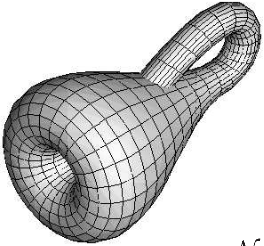
Notation: $\mathcal{S}_1$ is a *torus* (see on the left) – the surface of a donut.

$\mathcal{N}_1$ is a *projective plane*; if an open disc is removed from $\mathcal{N}_1$, then we get a Möbius b.

$\mathcal{N}_2$ is the *Klein bottle* (right); obtained by glueing two Möbius bands together.



$\mathcal{S}_1$



$\mathcal{N}_2$

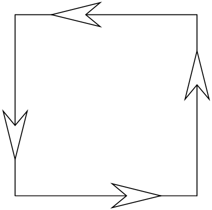
$\mathcal{S}_0$ and $\mathcal{S}_h$ are orientable surfaces, while all $\mathcal{N}_k$ are nonorientable.

It is instructive to see why no other surfaces can result by combined adding of handles and crosscaps:

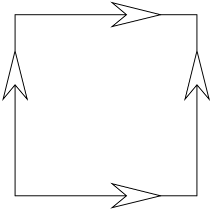
Lemma 11.2. *If a surface Σ is obtained from the sphere by adding $k > 2$ crosscaps and h handles, then Σ is homeomorphic to the one obtained by adding $k - 2$ crosscaps and $h + 1$ handles.*

Convention 11.3. As in classical topology, a surface can be represented as a polygon (disc) with prescribed oriented identification of its edges.

See the examples for the projective plane and the torus below.



$\mathcal{N}_1$



$\mathcal{S}_1$

11.2 Graph Embeddings in Surfaces

Definition 11.4. (cf. Definition 8.1) **An embedding** of a graph G on the surface Σ is a mapping which takes the vertices of G onto distinct points of Σ, and the edges of G onto simple arcs in Σ connecting their ends. No two edges are allowed to cross each other, and no edge is allowed to pass through another vertex. □

In order to smoothly translate properties of planar embeddings to other surfaces we moreover need:

Definition: An embedding of a graph G into Σ is *cellular* if each its face (without boundary) is homeomorphic to an open disc. □

Fact: A cellular embedding of 2-connected G is uniquely determined by its facial cycles, and this embedding also defines the underlying surface Σ up to homeomorphism.

In other words, Σ is "glued together" from the facial discs along shared edges. □

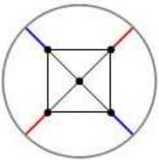
Proposition 11.5. *A cellular embedding of a graph G into an orientable surface is uniquely determined by the rotation scheme of its edges.*

A *rotation scheme* of a graph G determines the cyclic ordering of the edges around each of its vertices.

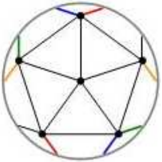
Embedded graphs

Recall that the complete graph K_5 is not planar.

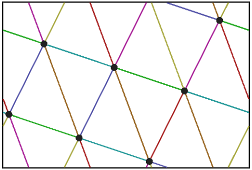
Proposition 11.6. *There exist embeddings of the complete graphs K_5 and K_6 in the projective plane, and of K_7 in the torus.*



K_5



K_6



K_7

On the other hand, K_7 does not embed in the Klein bottle. □

One can also straightforwardly extend Euler's formula as follows:

Theorem 11.7. *Let a cellular embedding of a nonempty graph G in Σ has f faces. Then*

$$|V(G)| + f - |E(G)| = \chi(\Sigma)$$

where $\chi(\Sigma)$ (the Euler characteristic of Σ) is $2-2h$ for $\Sigma = \mathcal{S}_h$ and $2-k$ for $\Sigma = \mathcal{N}_k$.

11.3 Embedding Obstructions, Minors

Similarly to algorithmic testing of planarity, embeddability in any other fixed surface can be tested in linear time by using the following strong result of Mohar:

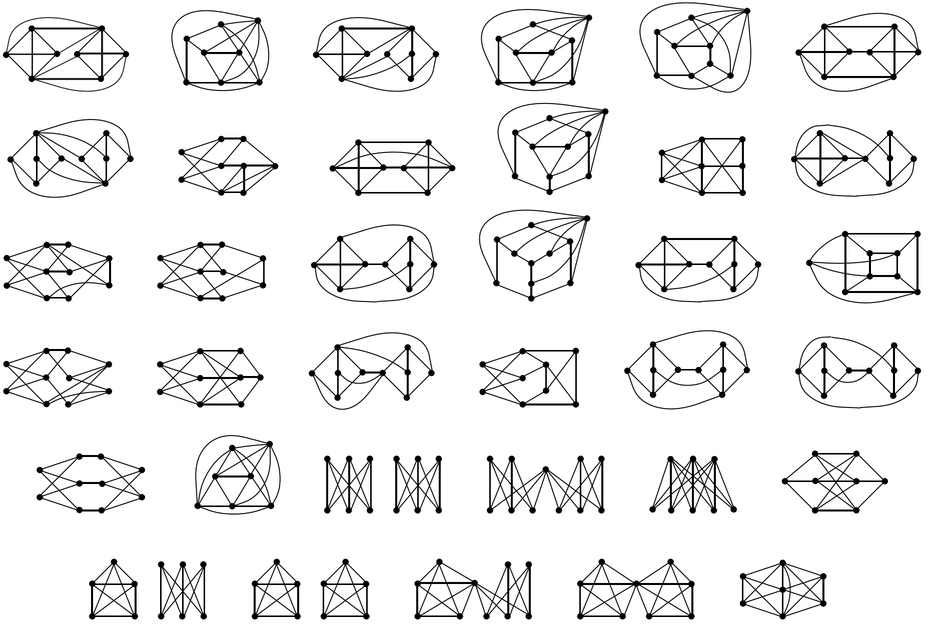
Theorem 11.8. *For every fixed surface Σ there is a linear-time algorithm that either finds an embedding in Σ, or outputs some minimal obstruction to embeddability in Σ.* □

The related problem to determine a smallest surface where the given graph embeds is already $\mathcal{NP}$-hard.

Notice that embeddability in a surface is preserved under taking subgraphs or even minors. Therefore, one would like to generalize the Kuratowski theorem to other surfaces. . . □

This is possible, but the lists of obstructions (forbidden minors) get very very large. In fact, such a complete list is known only in one case (Archdeacon):

Theorem 11.9. *A graph G embeds in the projective plane if, and only if, G contains no minor isomorphic to one of the following 35 graphs.*



Embeddings vs. Graph minors

Quite surprisingly, also the whole deep Graph minors theory of Robertson and Seymour builds upon graphs embedded in surfaces, though the final result itself does not seem to relate to embedded graphs. . . □

To explain this fact, we introduce one of the most important partial results of this theory.

Theorem 11.10. *Let H be a nonplanar graph. Then every graph G excluding a minor isomorphic to H has a tree-decomposition of the following property:*

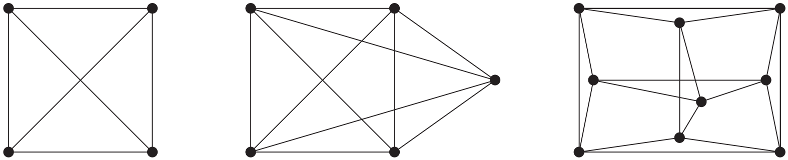
- *every bag induces a subgraph G_X of G such that G_X is, up to a bounded number of "local exceptions", embeddable in a surface Σ where H does not embed in.*

11.4 Crossing Number of a Graph

Another generalization of planar graphs aims at allowing edges to "cross" each other in a controlled way. This is based on the following definitions: □

Definition (another generalization of Definition 8.1):
A *drawing* of a graph G in the plane maps the vertices of G into distinct points in the plane, and the edges into simple arcs joining their endvertices. Moreover, no three edges can cross each other in the same point and no edge can pass through other vertex except its ends. □

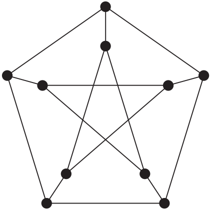
Example 11.11. *The following are valid drawings in the plane. Are the numbers of their edge crossings optimal?*

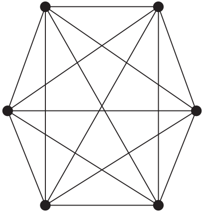


□

Definition 11.12. **The crossing number** of a graph G in the plane is the least number of pairwise edge crossings over all proper drawings of G in the plane. It is denoted by $cr(G)$. □

Example 11.13. *What are the crossing numbers of the following two graphs?*





□

Fact: Nobody knows for sure even the crossing numbers of complete and complete bipartite graphs!

On hardness of the crossing number problem

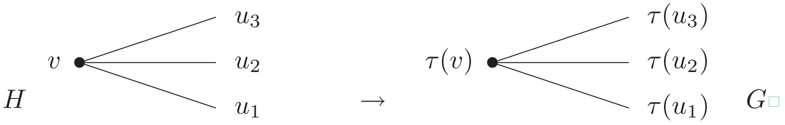
In fact, determining the crossing number of a given graph is a hopelessly hard problem.

- This problem is $\mathcal{NP}$-complete, even for cubic 3-connected graphs. □
- Though, for any fixed k, it can be tested in polynomial time whether $cr(G) \leq k$. Unfortunately, this is a totally impractical algorithm. □
- Nowadays, an involved b&b practical computing approach to the crossing number exists, see [Chimani et al.]. This one can, surprisingly, handle "real-world" graphs of < 100 vertices. □
- Still, Cabello and Mohar proved in 2010 that even for a planar graph G and a pair of vertices $u, v \in V(G)$, determining $cr(G + uv)$ is $\mathcal{NP}$-complete!

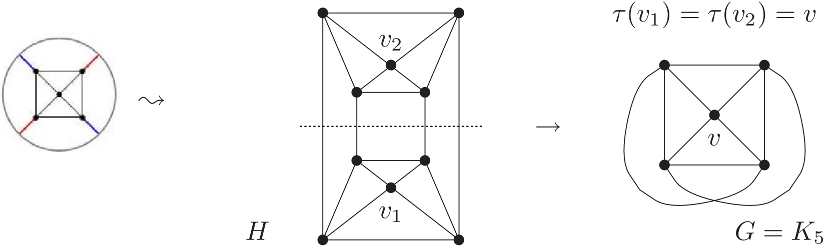
  The last result raises a worrying question; what else can be easy about computing the crossing number?

11.5 A note on Planar Covers and Emulators

Definition: A graph H *covers* a graph G if there exists a *locally-bijective* graph homomorphism $\tau : V(H) \to V(G)$, such that, for every vertex $v \in V(H)$, the neighbours of v in H are bijectively mapped onto the neighbours of $\tau(v)$ in G.



Proposition 11.14. *If G embeds in the projective plane, then the universal covering map of the projective plane by the sphere "lifts" G into a planar double cover H of G.*

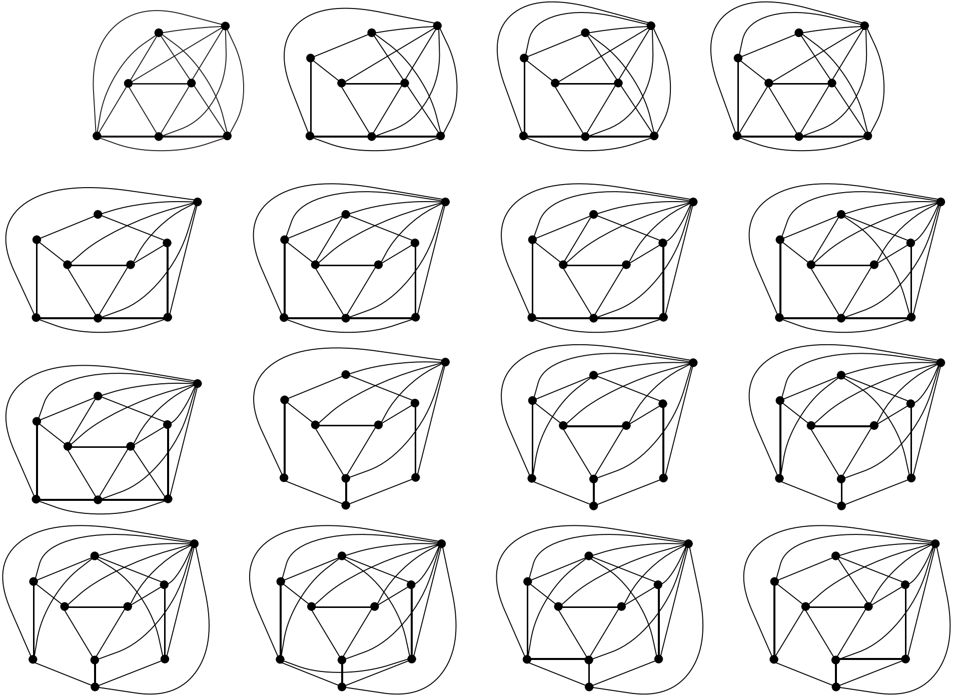


Negami's planar cover conjecture

Conjecture 11.15. A connected graph G has a cover by a finite planar graph if, and only if, G embeds in the projective plane. □

Regarding the (over) 20-years effort to solve this curious conjecture, the following steps have been gradually made by [Archdeacon, Fellows, Negami, Thomas, and the author]:

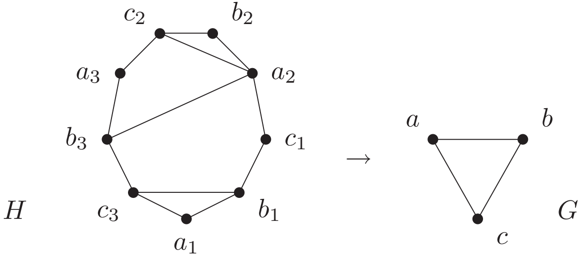
- The property of having a planar cover is preserved under taking subgr. / minors. □
- To prove Conjecture 11.15, it is enough to verify that none of the 32 connected forbidden minors from Theorem 11.9 has a finite planar cover. □
- The previous is quite easy for majority of those graphs.
- A few more specialized cases solved over the years. □
- Finally; if the graph $K_{1,2,2,2}$ had no finite planar cover, then Conjecture 11.15 would be true. □
- However, the best we know nowadays is that; there are at most 16 specific graphs (up to trivial modifications) for which Conjecture 11.15 might possibly fail.



On planar emulators

A small modification of the cover concept was independently suggested by Fellows.

Definition: A graph H *emulates* a graph G if there exists a *locally-surjective* graph homomorphism $\tau : V(H) \to V(G)$, such that, for every vertex $v \in V(H)$, the neighbours of v in H are surjectively mapped onto the neighbours of $\tau(v)$ in G.

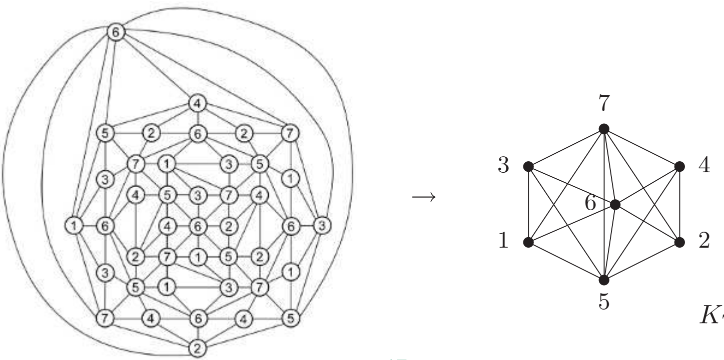


Fact: Being a planar cover is a special case of being a planar emulator.

Actually, where is any relevant difference between planar covers and planar emulators? How can the possibility of having "duplicate neighbours" help us in obtaining planar emulators (more edges make planarity harder!)...?

So, what is the relation between planar covers and planar emulators?

- Fellows conj. that any graph has a finite planar emulator iff it has such a cover.
- Though the conjecture seemed quite natural and nobody questioned it, more that 20 years later Rieck and Yamashita disproved this conjecture by providing two rather unexpected planar emulators of nonprojective graphs (2008).
- More planar emulators of nonprojective graphs have been found by Chimani and the author (2009). And then...
- In 2010, a student of MA010 (one like you) has found yet a new one:



$K_7 - C_4$