$$\begin{vmatrix} 1 & 0 & 0 & 2a \\ 0 & a & 1 & 0 \\ 0 & 1 & a & 0 \\ 2a & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a & 1 & 0 \\ 1 & a & 0 \\ 0 & 0 & 1 \end{vmatrix} - 2a \begin{vmatrix} 0 & 0 & 2a \\ a & 1 & 0 \\ 1 & a & 0 \end{vmatrix} =$$

$$= a^2 - 1 - 2a(2a^3 - 2a) =$$

$$= 5a^2 - 4a^4 - 1$$

$$a^5 \begin{vmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 \end{vmatrix} = -a^5 \begin{vmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{vmatrix} =$$

$$= -a^5 \begin{vmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{vmatrix} = -a^5(-4+5) = -a^5$$

$$\begin{vmatrix} a & & * & \\ & b & & \\ & & c & \\ 0 & & & d \end{vmatrix} = abcd$$

$(a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1) = \vec{o}$

$\Rightarrow \{(1,0,0), (0,1,0), (0,0,1)\}$ jsou generátory $\mathbb{R}^3$

navíc jsou lin. nezávislé

$= (0,0,0) \Rightarrow a = b = c = 0$

$$ax + by + z = 0 \qquad a, b \in \mathbb{R}$$
$$y + z = 0$$

Nechť (x_1, y_1, z_1) a (x_2, y_2, z_2) jsou řešením,

pak
$$ax_1 + by_1 + z_1 = 0 \quad \text{i} \quad ax_2 + by_2 + z_2 = 0$$
$$y_1 + z_1 = 0 \qquad\qquad y_2 + z_2 = 0$$

$$a(x_1 + x_2) + b(y_1 + y_2) + (z_1 + z_2) = 0$$
$$(y_1 + y_2) + (z_1 + z_2) = 0,$$

tedy $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$ je řešením dané soustavy.

$$y = -z$$

$$ax - bz + z = 0 \Rightarrow x = z\left(\frac{b-1}{a}\right)$$

$0 \neq a$

$\Rightarrow \left\langle \left(\frac{b-1}{a}, -1, 1\right) \right\rangle$ je prostor řešení dané soustavy, tj. $\left\{ \left(\frac{b-1}{a} \cdot t, -t, t\right), t \in \mathbb{R} \right\}$.

$$\langle \sin^2(x), \cos^2(x) \rangle = \{ a \cdot \sin^2(x) + b \cdot \cos^2(x), \; a, b \in \mathbb{R} \}$$

Konstantní funkce 1:

$$1 = \cos^2(x) + \sin^2(x)$$

má souřadnice $(1,1)$ v dané bázi.

Uvažme bázi $\{1, \sin^2 x\}$, v této bázi

Matrika prehoda T od baze $f = \{(1,0,1), (0,1,1), (1,-1,1)\}$ k stand. bazi:

$$T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow (T^*)^T = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix} \Rightarrow T^{-1} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$T \cdot \vec{v} = \vec{w} \Rightarrow \vec{v} = T^{-1} \vec{w}$$

$$T^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$a(1,0,1) + b(0,1,1) + c(1,-1,1) = (1,2,3)$$

$\mathbb{C}$ jako reálný vekt. prostor má dimenzi 2,
báze je např. $\{1, i\}$.

$$f = \{(1,1), (2,-1)\}$$

$$T = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \qquad T^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\frac{5}{3}(1+i) + \frac{1}{3}(2-i) = \\ = (2+i)$$

$$a(1+i) + b(2-i) = (2+i)$$

$$a + 2b = 2 \Rightarrow b+1+2b = 2 \Rightarrow b = \frac{1}{3} \Rightarrow$$
$$\Rightarrow a = \frac{5}{3}$$
$$a - b = 1 \Rightarrow a = b+1$$

Hledejme řešení tvaru $x_n = a^n$

$$a^{n+2} = 3a^{n+1} + 3a^n$$

$$a^2 = 3a + 3$$

$$a^2 - 3a - 3 = 0 \Rightarrow a_{1,2} = \frac{3 \pm \sqrt{21}}{2}$$

Prostor řešení dané rce je

$$\left\langle \left(\frac{3+\sqrt{21}}{2}\right)^n, \left(\frac{3-\sqrt{21}}{2}\right)^n \right\rangle,$$ tedy

lib. řešení je tvaru

$$a\left(\frac{3+\sqrt{21}}{2}\right)^n + b\left(\frac{3-\sqrt{21}}{2}\right)^n, \quad a, b \in \mathbb{R}$$

Dopočítejme x_0:

$$X_2 = 3X_1 + 3x_0 \Rightarrow 3x_0 = X_2 - 3X_1 = 0 \Rightarrow x_0 = 0$$

$$n=0: \; 0 = x_0 = a\left(\frac{3+\sqrt{21}}{2}\right)^0 + b\left(\frac{3-\sqrt{21}}{2}\right)^0 = a + b$$

$$a + b = 0 \Rightarrow b = -a$$

$$n=1: \quad X_1 = a\left(\frac{3+\sqrt{21}}{2}\right)^1 + b\left(\frac{3-\sqrt{21}}{2}\right)^1$$

$$1 = a\left(\frac{3+\sqrt{21}}{2}\right) + b\left(\frac{3-\sqrt{21}}{2}\right)$$

$$1 = a\left(\frac{3+\sqrt{21}}{2}\right) - a\left(\frac{3-\sqrt{21}}{2}\right) = \sqrt{21}\cdot a$$

$$\Rightarrow a = \frac{1}{\sqrt{21}}, \quad b = -\frac{1}{\sqrt{21}}$$

$$x_n = \frac{1}{\sqrt{21}}\left(\frac{3+\sqrt{21}}{2}\right)^n - \frac{1}{\sqrt{21}}\left(\frac{3-\sqrt{21}}{2}\right)^n.$$

$$x^2 = 2x - 2$$
$$x^2 - 2x + 2 = 0$$
$$x_{1,2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

Tj. posloupnosti $(1+i)^n$ a $(1-i)^n$ generují prostor všech řešení.

$$(1+i)^n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i \sin\frac{n\pi}{4}\right)$$
$$(1-i)^n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} - i \sin\frac{n\pi}{4}\right)$$
$$\Rightarrow \frac{1}{2}\left((1+i)^n + (1-i)^n\right) = \underline{(\sqrt{2})^n\left(\cos\frac{n\pi}{4}\right)}$$
$$\frac{1}{2i}\left((1+i)^n - (1-i)^n\right) = \underline{(\sqrt{2})^n\left(\sin\frac{n\pi}{4}\right)}$$

reálná báze prostoru řešení.

x_0: $\quad x_2 = 2x_1 - 2x_0 \Rightarrow x_0 = \frac{1}{2}(2x_1 - x_2) = 1 \quad \begin{pmatrix} x_1 = 2 \\ x_2 = 2 \end{pmatrix}$

Hledáme řešení ve tvaru $a(\sqrt{2})^n\left(\cos\frac{n\pi}{4}\right) + b(\sqrt{2})^n\left(\sin\frac{n\pi}{4}\right)$

$n=0: \quad a=1$

$n=1: \quad 2 = a+b \Rightarrow b=1$

$$x_n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4}\right) + (\sqrt{2})^n\left(\sin\frac{n\pi}{4}\right)$$

Vyjádřeme posloupnost jako lin. kombinaci $a(1+i)^n + b(1-i)^n$, kde $a, b \in \mathbb{C}$, $a = a_1 + i a_2$, $b = b_1 + i b_2$

$n=0: \quad a + b = 1 \Rightarrow a_1 + ia_2 + b_1 + ib_2 = 1 \Rightarrow a_1 = 1 - b_1, \ a_2 = -b_2$

$n=1: \quad a(1+i) + b(1-i) = 2$

$$(a_1 + ia_2)(1+i) + (1 - a_1 - a_2 i)(1-i) = 2$$

$\mathrm{Re}: \ a_1 - a_2 + 1 - a_1 - a_2 = 2 \Rightarrow a_2 = -\frac{1}{2}, \ a_1 = \frac{1}{2}$

$\mathrm{Im}: \ a_1 + a_2 + a_1 - 1 - a_2 = 0 \quad \Rightarrow b_1 = \frac{1}{2}, \ b_2 = \frac{1}{2}$

$$x_n = \left(\frac{1}{2} - \frac{1}{2} i\right)(1+i)^n + \left(\frac{1}{2} + \frac{1}{2} i\right)(1-i)^n$$