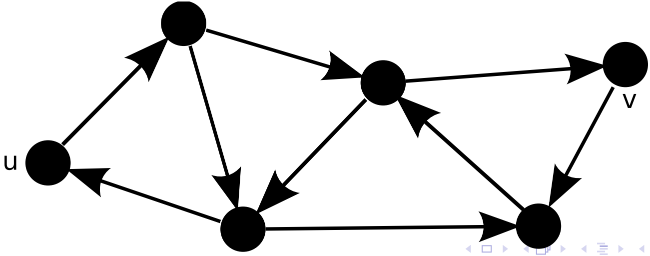
Shortest paths in graphs

Remarks from previous lectures:

- Path length in unweighted graph equals to edge count on the path
- Oriented distance ($\delta(u, v)$) between vertices u, v equals to the length of the shortest path from u to$v$
- In an oriented graph, distance between two vertices need not to be symmetrical ($\delta(u, v) \neq \delta(v, u)$ in general)

Figure: In this case $\delta(u, v) \neq \delta(v, u)$.



Distance in weighted graph

In real-world applications, graph edges are weighted – e. g., distances between cities, latency of network links.

Definition

Path length in weighted graph equals to sum of edge weights along the path.

- Distance between vertices is defined as the length of the shortest path between them.
- Negative-weight cycle potentially allows some or all distances in the graph to be any negative number.

By definition, the shortest paths do not contain any nonnegative-weight cycle.

Triangle inequality

The triangle inequality holds for a graph if and only if

$$\delta(u, w) \leq \delta(u, v) + \delta(v, w)$$

holds for any three vertices u, v, w in the graph.

Triangle inequality does not hold in general. A graph of the shortest (not direct) distances between cities is the real-world example in which the inequality holds.

Dijkstra's algorithm

- Well-known algorithm for finding single-source shortest paths.
- Solves the problem for both directed and undirected graphs.
- Computes shortest paths from single source vertex to all others.
- Requires non-negative weights of all edges (not only cycles).
- Linear space complexity.
- Time complexity depends on chosen data structure.

Dijkstra's algorithm

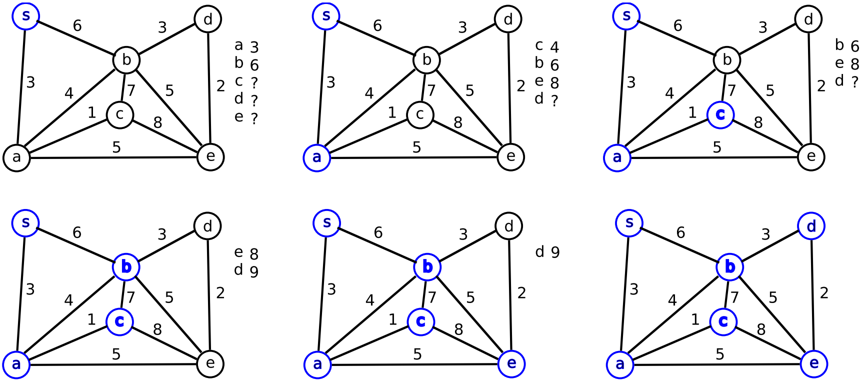
- Denote source vertex as s.
- For each vertex v in a graph, `d[v]` equals to length the shortest path from s to v found so far.
  - Initially, `d[s]` $= 0$ for source vertex and `d[v]` $= \infty$ for the others.
  - Upon completion, `d[v]` equals to length of the shortest path in the graph if it exists, or ∞ otherwise.
- `p[v]` stores the direct predecessor of vertex v on the shortest path from s found so far.
  - Initially, `p[v]` is undefined for all vertices except s.
  - Upon completion, the shortest path to v is the sequence `s, p[...p[v]...], ... p[p[v]], p[v], v`.

Dijkstra's algorithm

- Vertices are split into two disjoint sets:
  - S contains exactly those vertices, for which the shortest paths has already been computed and stored in `d[v]`.
  - Q contains all other vertices.
- The vertices of set Q are stored in a priority queue.
  - The vertex with the lowest value of `d[u]` has the highest priority. The `d[u]` already stores length of the shortest path to u.
- Following steps are taken in each iteration:
  - Remove the vertex u from the queue head.
  - Move the vertex u from Q to S.
  - Relax all edges (u, v) going out from u to any v in Q:
    - If `d[v]` $>$ `d[u]` $+w(u, v)$, update `d[v]`.
  - $w(u, v)$ denotes weight of the edge (u, v).

Dijkstra's algorithm – example

Figure: Vertices in the set S are marked blue. Content of the priority queue is depicted to the right of the graph (head on top).



Dijkstra's algorithm – animations & illustrations

- Animation on an example graph
  - http://www.unf.edu/~wkloster/foundations/DijkstraApplet/DijkstraApplet.htm
- commented computation
  - http://www.youtube.com/watch?v=8Ls1RqHCOPw
- computation allowing to input your own graph
  - http://www.cse.yorku.ca/~aaw/HFHuang/DijkstraStart.html
- illustration of a computation
  - http://www.animal.ahrgr.de/showAnimationDetails.php3?lang=en&anim=16

Dijkstra's algorithm – time complexity

Let's denote $n = |V|, m = |E|$.

- Initialization is linear w. r. t. number of vertices.
- Each edge is traversed exactly once or twice (in case of oriented graph).
- Main loop is run n-times, hence
- there are n delete operations on the priority queue.
- Complexity of the delete operations depends on chosen data structure:
  - **Array, vertex list** – deletion can be done in linear time, complexity of the whole algorithm is therefore in $\mathcal{O}(n^2 + m)$.
  - **Binary heap** – deletion requires $\mathcal{O}(log(n))$ time. Moreover, each edge relaxation may require heap update ($\mathcal{O}(log(n))$), overall complexity is in $\mathcal{O}((n + m)log(n))$.
  - **Fibonacci's heap** – complexity of the deletion is the same as in the case of binary heap, however update on relaxation runs in constant time – overall complexity is in $\mathcal{O}(m + nlog(n))$. `http://en.wikipedia.org/wiki/Fibonacci_heap`

Dijkstra's algorithm – application in networks

Link-state routing protocols make use of the Dijkstra's algorithm.

- Each active elements broadcasts its neighbors list periodically
- Neighbors list are forwarded through the network to all active elements
- Each active element calculates a shortest paths tree to all other AEs independently
- Risk of loops in routing tables

OSPF and IS-IS are the most widespread link-state protocols. They both use the Dijkstra's algorithm.

Floyd-Warshall's algorithm

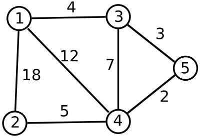
- Computes shortest paths between each pair of vertices.
- The algorithm works with negative-weight edges correctly, however, negative-weight cycles may lead to incorrect solution.
- The shortest (so far) known distance between any two vertices is being improved gradually.
- In each step, a set of vertices which may lie on the shortest paths is defined.
- Each iteration introduces a new vertex into this set.
- In each one of n iterations, shortest paths between all n^2 pairs of vertices are updated. The time complexity therefore equals to $\mathcal{O}(n^3)$.
- The space complexity is $\mathcal{O}(n^2)$.

Floyd-Warshall's algorithm

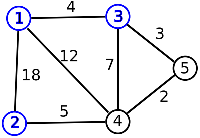
- Let the vertices be numbered as $1 \ldots n$.
- At first, only single-edge paths are considered. Afterwards, the algorithm searches for paths traversing through vertex 1. Subsequently, paths using vertices 1 and 2, etc.
- Between any pair of vertices u, v, a shortest path using vertices $1 \ldots k$ is known in $(k+1)i^{th}$ iteration.
- There are two possibilities for the shortest path (which uses vertices $1 \ldots k+1$) between these two vertices:
  - It uses only the $1 \ldots k$ vertices.
  - It traverses vertices $1 \ldots k$ from u to vertex $k+1$ and then ends in v.
- Upon completion, shortest paths using all vertices in the graph are computed.

Floyd-Warshall's algorithm – an example

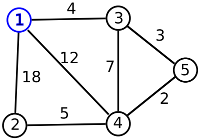
Figure: Vertices which may be used for shortest paths are highlighted. Shortest paths computed so far are stored in the matrix.



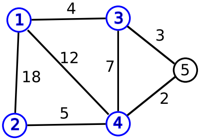
	1	2	3	4	5
1	0	18	4	12	?
2	18	0	?	5	?
3	4	?	0	7	3
4	12	5	7	0	2
5	?	?	3	2	0



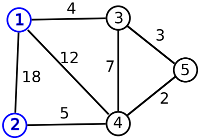
	1	2	3	4	5
1	0	18	4	11	7
2	18	0	22	5	25
3	4	22	0	7	3
4	11	5	7	0	2
5	7	25	3	2	0



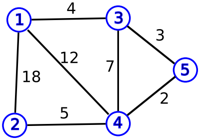
	1	2	3	4	5
1	0	18	4	12	?
2	18	0	22	5	?
3	4	22	0	7	3
4	12	5	7	0	2
5	?	?	3	2	0



	1	2	3	4	5
1	0	16	4	11	7
2	16	0	12	5	7
3	4	12	0	7	3
4	11	5	7	0	2
5	7	7	3	2	0



	1	2	3	4	5
1	0	18	4	12	?
2	18	0	22	5	?
3	4	22	0	7	3
4	12	5	7	0	2
5	?	?	3	2	0



	1	2	3	4	5
1	0	14	4	9	7
2	14	0	10	5	7
3	4	10	0	5	3
4	9	5	5	0	2
5	7	7	3	2	0

Distributed Floyd-Warshall's algorithm

Floyd-Warshall's algorithm can be easily applied in distributed environment – among autonomous units, which communicate only through message sending

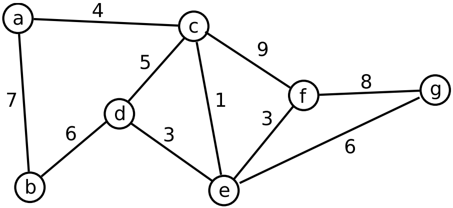
- Each vertex computes shortest paths to all other graph vertices
- Initially, only path to neighbours is known
- Similarly to the sequential case, each iteration adds single vertex which can be included in the paths
- Added vertex broadcasts its distances table to all other vertices in each iteration
- The other vertices update their distances and shortest paths according to the received table

Distributed Floyd-Warshall's algorithm

- It is crucial for correctness of the algorithm that all vertices choose the same vertex in each iteration.
- Algorithm is inefficient in terms of transferred data amount. If `d[v]` $= \infty$ holds for selected vertex v in any vertex, its paths are not updated at all, hence it does not need to receive any distance tables in the current iteration.
- Before broadcasting distance table, vertices may signal to each other, which of them should receive the table $\Rightarrow$ Toueg's algorithm.
- Further information:
    - Ajay D. Kshemkalyani, Mukesh Singhal. *Distributed Computing: Principles, Algorithms, and Systems*. Cambridge University Press, 2008. Pp. 151-155

Excercises

1. Calculate shortest paths in the graph below using Dijkstra's and Floyd-Warshall's algorithm.



2. Propose an implementation of the Floyd-Warshall's algorithm (Toueg's algorithm). Consider, that vertices can transmit messages only along graph edges (broadcasting is implemented by forwarding).

Excercises

3. Why doesn't Dijkstra's algorithm work correctly on graphs with negative-weight edges? What are the possible outcomes when it is run on such graph?