a setting, a rule of inference is specified by giving a list of hypotheses and a conclusion that may be drawn from them. The standard rule of modus ponens (I.7.2) concludes β from the hypotheses α and $\alpha \to \beta$. An appropriate style for describing this rule is to write the hypotheses in a list above the line and the conclusion below:

$$\frac{\alpha, \alpha \to \beta}{\beta}.$$

In this notation, the axioms are simply rules without hypotheses such as in I.7.1(i):

$$\frac{}{(\alpha \to (\beta \to \alpha))}.$$

The crucial extension of such a system to nonmonotonic logic is to add restraints to the deduction. In addition to knowing each proposition in the set of hypotheses, it may be necessary to not know (believe, have a proof of, have already established, etc.) each of some other collection of propositions in order to draw the conclusion permitted by a given rule. The notation for this situation is to list the usual kind of premises first and then, separated by a colon, follow them with the list of restraints. The restraints are the propositions that the rule requires us not to know (believe, etc.). Thus we read the rule

$$\frac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_m}{\gamma}$$

as saying that if $\alpha_1, \ldots, \alpha_n$ are known (proven, established) and $\beta_1, \ldots, \beta_m$ are not, then we may conclude that we know (can prove or establish) γ.

Definition 7.1 (*Nonmonotonic formal systems*): Let U be a set (of propositional letters).

(i) A *nonmonotonic rule of inference* is a triple $\langle P, G, \varphi \rangle$ where $P = \{\alpha_1, \ldots, \alpha_n\}$ and $G = \{\beta_1, \ldots, \beta_m\}$ are finite lists of elements of U and $\varphi \in U$. Each such rule is written in the form

$$r = \frac{\alpha_1, \ldots, \alpha_n : \beta_1, \ldots, \beta_m}{\varphi}.$$

We call $\alpha_1, \ldots, \alpha_n$ the *premises* of the rule r and $\beta_1, \ldots, \beta_m$ its *restraints*. Note that either P or G or both may be empty.

(ii) If $P = G = \emptyset$, then the rule r is called an *axiom*.

(iii) A *nonmonotonic formal system* is a pair $\langle U, N \rangle$ where U is a nonempty set (of propositional letters) and N is a set of nonmonotonic rules.

(iv) A subset S of U is *deductively closed* in the system $\langle U, N \rangle$ if, for each rule r of N such that all the premises $\alpha_1, \ldots, \alpha_n$ of r are in S and none of its restraints $\beta_1, \ldots, \beta_m$ are in S, the conclusion φ of r is in S.

The essence of the nonmonotonicity of a formal system is that the deductively closed sets are not in general closed under arbitrary intersections. Thus there is, in general, no deductive closure of a set I of propositional letters, i.e., no least set $S \supseteq I$ which is deductively closed. The intersection of a decreasing sequence of deductively closed sets is, however, deductively closed (Exercise 1) and so there is always (at least one) minimal deductively closed subset of U (Exercise 2).

The deductively closed sets containing I can be viewed as the rational points of view possible in a given system when one assumes all the elements of I to be true. Each one expresses a set of beliefs that is closed under all the rules. There may, however, be many such points of view that are mutually contradictory. The intersection of all deductively closed sets containing I represents the information common to all such rational points of view. It is often called the set of *secured consequences* of I or the *skeptical reasoning* associated with the system and I.

Example 7.2: Let $U = \{\alpha, \beta, \gamma\}$ and let

$$r_1 = \frac{}{\alpha} \qquad r_3 = \frac{\alpha : \beta}{\gamma}$$
$$r_2 = \frac{\alpha : \beta}{\beta} \qquad r_4 = \frac{\alpha : \gamma}{\beta}.$$

(i) Let $N_1 = \{r_1, r_2\}$. There is only one minimal deductively closed set for $\langle U, N_1 \rangle : S = \{\alpha, \beta\}$. S is then the set of secured consequences of $\langle U, N_1 \rangle$.

(ii) Let $N_2 = \{r_1, r_3, r_4\}$. There are two minimal deductively closed sets for $\langle U, N_2 \rangle : S_1 = \{\alpha, \beta\}$ and $S_2 = \{\alpha, \gamma\}$. $S = \{\alpha\}$ is then the set of secured consequences of $\langle U, N_2 \rangle$. In this case the set of secured consequences is not deductively closed.

The analog in nonmonotonic logic of a classical deduction from premises I involves a parameter S for the set of propositions we are assuming we do not know. We use this notion of deduction to characterize the *extensions* of a nonmonotonic system that are analogous to the set of consequences of a monotonic system.

Definition 7.3: Let $\langle U, N \rangle$ be a nonmonotonic formal system and let $S, I \subseteq U$. An *S-deduction of φ from I in $\langle U, N \rangle$* is a finite sequence $\varphi_1, \ldots, \varphi_k$ such that $\varphi = \varphi_k$ and, for all $i \leq k$, φ_i is either in I, an axiom of $\langle U, N \rangle$ or the conclusion of a rule $r \in N$ all of whose premises are included among $\varphi_1, \ldots, \varphi_{i-1}$ and all of whose restraints are contained in $U - S$. In this situation φ is called an *S-consequence of I* and we denote by $C_S(I)$ the set of all S–consequences of I.

Note that the role of S in the above definitions is to prevent applications of rules with any restraint in S; it does not contribute any members of U directly to $C_S(I)$. Indeed, $C_S(I)$ may not contain S and may not be deductively closed.