$$p(1,2).$$
$$q(1) :- \ p(1,1), \neg q(1).$$
$$q(1) :- \ p(1,2), \neg q(2).$$
$$q(2) :- \ p(2,1), \neg q(1).$$
$$q(2) :- \ p(2,2), \neg q(2).$$

We now consider the subset $M = \{q(1)\}$ of the Herbrand universe. P_M is then

$$p(1,2).$$
$$q(1) :- \ p(1,2).$$
$$q(2) :- \ p(2,2).$$

The minimal Herbrand model of P_M is $\{p(1,2), q(1)\} \neq M$. Thus, M is not a stable model of P. Indeed, any set M not containing $p(1,2)$ is not a model of P and so by Theorem 7.9 not a stable model of P.

Next, we consider the two minimal Herbrand models M_1 and M_2 of the original program Q. We claim that M_1 is stable but not M_2. First, P_{M_1} is

$$p(1,2).$$
$$q(1) :- \ p(1,2).$$
$$q(2) :- \ p(2,2).$$

The minimal model of this program is clearly M_1 which is therefore stable. On the other hand, P_{M_2} is

$$p(1,2).$$
$$q(1) :- \ p(1,1).$$
$$q(2) :- \ p(2,1).$$

Its minimal model is $\{p(1,2)\} \neq M_2$. Thus M_2 is not stable.

In fact, M_1 is the only stable model of P (Exercise 7) and so the stable model is the "right" one from the viewpoint of negation as failure.

A direct and precise connection of stable models with nonmonotonic formal systems is provided by the Theorem 7.12. We continue with the notation introduced above and begin with a lemma.

Lemma 7.11: *If M' is a model of P_M, then $M' \supseteq C_M(\emptyset)$.*

Proof: Suppose that M' is a model of P_M. We prove by induction on the length of M–deductions that every member of $C_M(\emptyset)$ is contained in M'. Consider any M–deduction $\varphi_1, \ldots, \varphi_k, p$ (from $\emptyset$) and suppose the rule applied at the last step of this deduction to conclude p is $tr(C)$ for some clause C in P. By induction, we may assume that $\varphi_i \in M'$ for every $1 \leq i \leq k$ and so every premise q_i of $tr(C)$ is in M'. As this is an M–deduction, no restraint s_j of $tr(C)$ is in M. By definition then, $p :- \ q_1, \ldots, q_n$ is one of the clauses of P_M. As M' is a model of P_M, $p \in M'$ as required. □

Theorem 7.12: *A subset M of U is a stable model of P if and only if it is an extension of $tr(P)$.*

Proof: Suppose that M is an extension of $\langle U, tr(P) \rangle$. First, we claim that M is a model of P_M. Consider any clause $p :- \ q_1, \ldots, q_n$ in P_M such that $q_1, \ldots, q_n \in M$. By the definition of P_M, there is a clause $C = p :- \ q_1, \ldots, q_n, \neg s_1, \ldots, s_m$ in P with no s_j in M. Thus there is a rule $tr(C)$ in $tr(P)$ with all its premises in M and none of its restraints in M. As extensions are deductively closed by Proposition 7.5, $q \in M$ as required. Next, we must prove that no M' strictly contained in M is a model of P_M. As $M = C_M(\emptyset)$, this is immediate from Lemma 7.11.

For the converse, suppose that M is a minimal Herbrand model of P_M. We first note that, by Lemma 7.11, $M \supseteq C_M(\emptyset)$. By the minimality assumption on M, it suffices to prove that $C_M(\emptyset)$ is a model of P_M to conclude that $M = C_M(\emptyset)$ as required. Consider, therefore, any clause $p :- \ q_1, \ldots, q_n$ in P_M with all the q_i in $C_M(\emptyset)$. There is then an M–deduction $\varphi_1, \ldots, \varphi_k$ containing all of the q_i. By definition of P_M, there is a clause $C = p :- \ q_1, \ldots, q_n, \neg s_1, \ldots, \neg s_m$ in P with none of the s_j in M and so a rule $tr(C)$ in $tr(P)$ with all its premises in $C_M(\emptyset)$. We may thus form an M–deduction with p as the consequence. So $p \in C_M(\emptyset)$ as required. □

Gelfond and Lifschitz show that certain classes of programs with properties such as those considered in §4 have unique stable models and propose the term stable model semantics for such programs. The special case of a unique stable model is certainly of particular interest. From the viewpoint of nonmonotonic logic, however, all the extensions of $tr(P)$ are equally good candidates for models of the system.

Exercises

1. Let $S_1 \supseteq S_2 \supseteq \ldots$ be a nested sequence of deductively closed sets for a nonmonotonic system $\langle U, N \rangle$. Prove that $\cap S_i$ is deductively closed.

2. A version of Zorn's lemma (see Theorem VI.10.2 and Exercise VI.10.2) states that any nonempty family of sets closed under the intersection of downwardly nested sequences has a minimal element. Use it and Exercise 1 to prove that every nonmonotonic formal system has a minimal deductively closed subset.

3. Prove that the operation $C_S(I)$ is monotonic in I and antimonotonic in S, that is if $I \subseteq J$, then $C_S(I) \subseteq C_S(J)$ and if $S \subseteq T$, then $C_S(I) \supseteq C_T(I)$.

4. Prove that, if S is an extension of I, then S is a minimal deductively closed superset of I and for every J such that $I \subseteq J \subseteq S$ we have $C_S(J) = S$.