# Modal & temporal logic

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adopted by Juraj Jurco from slides of Rajeev Gore

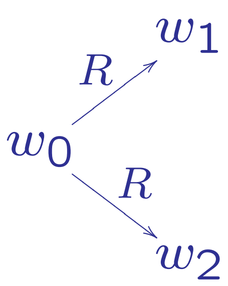
# Kripke Semantics for Logical Consequence

Given some model $\langle W, R, \vartheta \rangle$ and some $w \in W$, we compute the truth value of a non-atomic formula by recursion on its shape:

$$\vartheta(w, \langle\rangle\varphi) = \begin{cases} \mathrm{t} & \vartheta(v,\varphi) = \mathrm{t} \text{ for some } v \in W \text{ with } wRv \\ \mathrm{f} & \text{otherwise} \end{cases}$$

$$\vartheta(w, []\varphi) = \begin{cases} \mathrm{t} & \vartheta(v,\varphi) = \mathrm{t} \text{ for every } v \in W \text{ with } wRv \\ \mathrm{f} & \text{otherwise} \end{cases}$$

Example: If $W = \{w_0, w_1, w_2\}$ and $R = \{(w_0, w_1), (w_0, w_2)\}$ and $\vartheta(w_1, p_3) = \mathrm{t}$ then $\langle W, R, \vartheta \rangle$ is a Kripke model as pictured below:



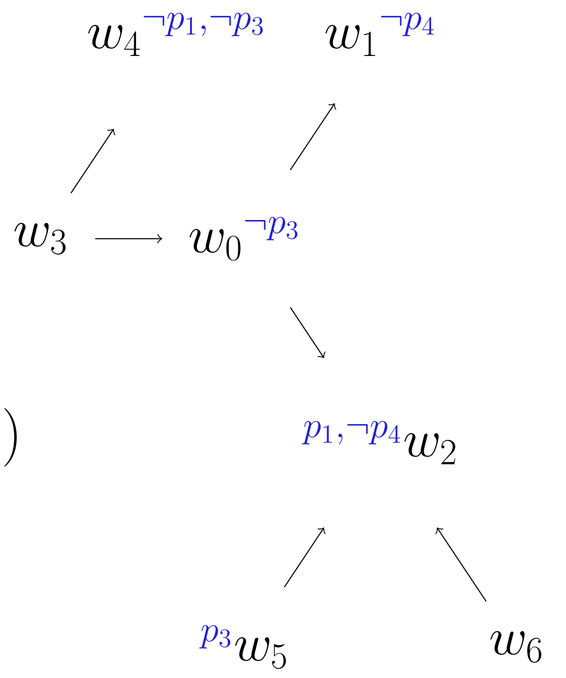

$$\begin{aligned} \vartheta(w_0, \langle\rangle p_3) &= \mathrm{t} \\ \vartheta(w_0, []p_3) &= \mathrm{f} \\ \vartheta(w_1, []p_1) &= \mathrm{t} \\ \vartheta(w_1, []\neg p_1) &= \mathrm{t} \\ \vartheta(w_0, \langle\rangle[]p_1) &= \mathrm{t} \end{aligned}$$

Intuition: truth of modalities depends on underlying $R$ (not truth functional)

# Modal & temporal logic - examples

$W = \{w_0, w_1, w_2, w_3, w_4, w_5, w_6\}$,
$R = \{(w_0, w_1), (w_0, w_2), (w_3, w_0),$
$(w_3, w_4), (w_5, w_2), (w_6, w_2)\}$

$\vartheta$ – *set by graph*




$(w_4, \langle\rangle\, []p_2) = f;$ $(no\ following\ world\ exists)$
$(w_0, \langle\rangle\, []\neg p_1) = t;$
$(w_3, \langle\rangle\, []\neg p_4) = t; (true\ for\ w_4, w_0)$
$(w_0, []\, \langle\rangle\, p_1) = f;$ $(false\ for\ w_1, w_2)$
$(w_3, \langle\rangle\, p_3) = f;$

# Semantic Forcing Relation $\Vdash$ and its negation $\nVdash$

Let $\mathcal{K}$ be the class of all Kripke models, and $\mathcal{M} = \langle W, R, \vartheta \rangle$ a Kripke model

Let $\mathfrak{K}$ be the class of all Kripke frames and let $\mathfrak{F}$ be a Kripke frame

Let $\Gamma$ be a set of formulae, and $\varphi$ be a formula

| Forces | We say | We write | When | $\bullet \nVdash \varphi$ |
|---|---|---|---|---|
| in a world | $w$ forces $\varphi$ | $w \Vdash \varphi$ | $\vartheta(w, \varphi) = \mathbf{t}$ | $\vartheta(w, \varphi) = \mathbf{f}$ |
| in a model | $\mathcal{M}$ forces $\varphi$ | $\mathcal{M} \Vdash \varphi$ | $\forall w \in W. w \Vdash \varphi$ | $\exists w \in W. w \nVdash \varphi$ |
| in a frame | $\mathfrak{F}$ forces $\varphi$ | $\mathfrak{F} \Vdash \varphi$ | $\forall \vartheta. \langle \mathfrak{F}, \vartheta \rangle \Vdash \varphi$ | $\exists \vartheta. \langle \mathfrak{F}, \vartheta \rangle \nVdash \varphi$ |

Classicality: either $\bullet \Vdash \varphi$ or else $\bullet \nVdash \varphi$ holds for $\bullet \in \{w, \mathcal{M}, \mathfrak{F}\}$

Exercise: Work out the negation of each fully e.g. $\mathcal{M} \nVdash \varphi$ is $\exists w \in W. w \Vdash \neg\varphi$

Either $w \Vdash \varphi$ or else $w \Vdash \neg\varphi$ holds (Lemma 1)

But this does not apply to all: e.g. either $\mathcal{M} \Vdash \varphi$ or else $\mathcal{M} \Vdash \neg\varphi$ is rarely true.

$W \Vdash \varphi$ meaning “every frame built out of given $W$ forces $\varphi$” is not interesting

# Lecture 5: Tense and Temporal Logics

Tense Logics: interpret $[]\varphi$ as "$\varphi$ is true always in the future".

$W$ represents moments of time

$R$ captures the flow of time

Temporal Logics: similar, but use a more expressive binary modality $\varphi \mathcal{U} \psi$ to capture "$\varphi$ is true at all time points from now until $\psi$ becomes true".

Shall look at Syntax, Semantics, Hilbert and Tableau Calculi.

# Tense Logics: Syntax and Semantics

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$

Formulae: $\varphi ::= p \mid \neg\varphi \mid \langle F\rangle\varphi \mid [F]\varphi \mid \langle P\rangle\varphi \mid [P]\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi$

Boolean connectives interpreted as for modal logic.

Given some Kripke model $\langle W, R, \vartheta\rangle$ and some $w \in W$, we compute the truth value of a non-atomic formula by recursion on its shape:

$$\vartheta(w, \langle F\rangle\varphi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at some } v \in W \text{ with } wRv \\ \mathrm{f} & \text{otherwise} \end{cases}$$

$$\vartheta(w, [F]\varphi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at every } v \in W \text{ with } wRv \\ \mathrm{f} & \text{otherwise} \end{cases}$$

$$\vartheta(w, \langle P\rangle\varphi) = \begin{cases} \mathrm{t} & \text{if } \vartheta(v,\varphi) = \mathrm{t} \text{ at some } v \in W \text{ with } vRw \\ \mathrm{f} & \text{otherwise} \end{cases}$$

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# Tense Logics: Syntax and Semantics

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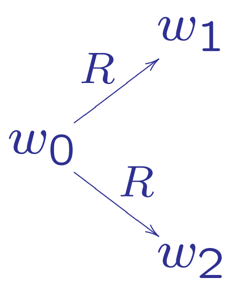
Example: If $W = \{w_0, w_1, w_2\}$ and $R = \{(w_0, w_1), (w_0, w_2)\}$ and $\vartheta(w_1, p_3) = \mathrm{t}$ then $\langle W, R, \vartheta\rangle$ is a Kripke model as pictured below:

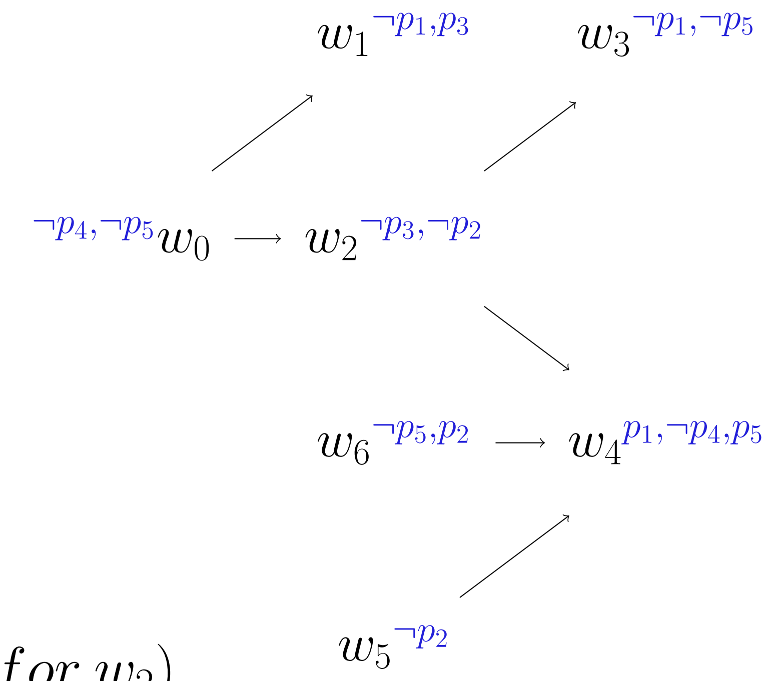



$$\begin{aligned} \vartheta(w_0, \langle F\rangle p_3) &= \mathrm{t} \\ \vartheta(w_2, \langle P\rangle\langle F\rangle p_3) &= \mathrm{t} \\ \vartheta(w_0, [P]p_1) &= \mathrm{t} \end{aligned}$$

# Modal & temporal logic - examples

$W = \{w_0, w_1, w_2, w_3, w_4, w_5, w_6\}$,
$R = \{(w_0, w_1), (w_0, w_2), (w_2, w_3),$
$(w_2, w_4), (w_6, w_4), (w_5, w_4)\}$

$\vartheta$ – *set by graph*




$(w_4, [P]p_2) = f;$
$(w_2, \langle P \rangle \langle F \rangle p_3) = t;$
$(w_4, \langle P \rangle [F]p_1) = t;$ $(true\ for\ w_5, w_6)$
$(w_4, \langle P \rangle [P]p_4) = t;$ $(true\ for\ w_5, w_6; false\ for\ w_2)$
$(w_0, [F] \langle F \rangle p_5) = f;$ $(true\ for\ w_2; false\ for\ w_1)$

# Different Models of Time

Arbitrary Time: $\mathbf{K_t}$

Reflexive Time: $\varphi \rightarrow \langle F \rangle \varphi$

Transitive Time: $\langle F \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi$

Dense Time: $\langle F \rangle \varphi \rightarrow \langle F \rangle \langle F \rangle \varphi$

Never Ending Time: $[F]\varphi \rightarrow \langle F \rangle \varphi$

Backward Linear: $\langle F \rangle \langle P \rangle \varphi \rightarrow \langle P \rangle \varphi \vee \varphi \vee \langle F \rangle \varphi$

Forward Linear: $\langle P \rangle \langle F \rangle \varphi \rightarrow \langle F \rangle \varphi \vee \varphi \vee \langle P \rangle \varphi$

Tableau Calculi also exist but require even more complex loop detection often called "dynamic blocking".

Discrete $\langle \mathbb{Z}, < \rangle$ , Rational $\langle \mathbb{Q}, < \rangle$, Real $\langle \mathbb{R}, < \rangle$ linear and non-reflexive models of time also possible: see Goldblatt.

Tableau-like calculi exist: see Mosaic Method

# PLTL: Propositional Linear Temporal Logic

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$

Formulae: $\varphi ::= p \mid \neg\varphi \mid \bigoplus\varphi \mid [F]\varphi \mid \langle F\rangle\varphi \mid \varphi\mathcal{U}\psi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$

Boolean connectives interpreted as for modal logic.

Linear Time Kripke Model: $\langle S, \sigma, R, \vartheta\rangle$

$S$**:** non-empty set of states

$\sigma$**:** $\mathbb{N} \rightarrow S$ enumerates $S$ as sequence $\sigma_0, \sigma_1, \cdots$ with repetitions when $S$ finite

$\vartheta$**:** $S \times Atm \mapsto \{\mathbf{t}, \mathbf{f}\}$

$R$**:** is a binary relation over $S$

**Condition:** $R = \sigma^*$ ($R$ is the reflexive and transitive closure of $\sigma$)

# Semantics of PLTL

$$\vartheta(s_i, \bigoplus\varphi) = \begin{cases} \mathbf{t} & \text{if } \vartheta(s_{i+1}, \varphi) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$\vartheta(s_i, \langle F\rangle\varphi) = \begin{cases} \mathbf{t} & \text{if } \vartheta(s_j, \varphi) = \mathbf{t} \text{ for some } j \geq i \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$\vartheta(s_i, [F]\varphi) = \begin{cases} \mathbf{t} & \text{if } \vartheta(s_j, \varphi) = \mathbf{t} \text{ for all } j \geq i \\ \mathbf{f} & \text{otherwise} \end{cases}$$

$$\vartheta(s_i, \varphi\mathcal{U}\psi) = \begin{cases} \mathbf{t} & \text{if } \exists k \geq i.\vartheta(s_k, \psi) = \mathbf{t} \,\&\, \forall j.i \leq j < k \Rightarrow \vartheta(s_j, \varphi) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

| | | | | | |
|---|---|---|---|---|---|
| $s_i$ | $s_{i+1}$ | $\cdots$ | $s_j$ | $\cdots$ | $s_k$ |
| $p\mathcal{U}q$ | $p, \neg q$ | $\cdots$ | $p, \neg q$ | $\cdots$ | $q$ |

Note: when $k \neq i$, the state $s_k$ is the first state after $s_i$ where $q$ is true.

# Semantics of PLTL

$$\vartheta(s_i, \bigoplus\varphi) = \begin{cases} \mathbf{t} & \text{if } \vartheta(s_{i+1}, \varphi) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

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$$\vartheta(s_i, \varphi\mathcal{U}\psi) = \begin{cases} \mathbf{t} & \text{if } \exists k \geq i.\vartheta(s_k, \psi) = \mathbf{t} \ \& \ \forall j.i \leq j < k \Rightarrow \vartheta(s_j, \varphi) = \mathbf{t} \\ \mathbf{f} & \text{otherwise} \end{cases}$$

| $s_i$ | $s_{i+1}$ | $\cdots$ | $s_j$ | $\cdots$ | $s_k$ | |
|---|---|---|---|---|---|---|
| $\neg(p\mathcal{U}q), \neg q$ | $\neg q$ | $\cdots$ | $\neg q$ | $\cdots$ | $\neg q$ | $q$ is always false, or |
| $\neg(p\mathcal{U}q)$ | $\neg q$ | $\cdots$ | $\neg p, \neg q$ | $\cdots$ | $q$ | $p$ false before $q$ true |

Note: when $k \neq i$, the state $s_k$ is the first state after $s_i$ where $q$ is true. And $p$ is false in some $s_j$ before state $s_k$.

# Modal & temporal logic - examples

$$W = \{w_{1\ldots5}\}; R \,\&\, \vartheta - Set\ by\ graph$$

$$^{p,\neg q,s}w_1 \longrightarrow {}^{p,\neg q}w_2 \longrightarrow {}^{p,\neg q}w_3 \longrightarrow {}^{\neg p,\neg q}w_4 \longrightarrow {}^{q,s}w_5$$

$$\vartheta(w_1, \neg q \mathcal{U} \neg p) = t$$
$$\vartheta(w_1, \neg q \mathcal{U} s) = t$$
$$\vartheta(w_1, \neg s \mathcal{U} q) = f$$
$$\vartheta(w_2, \oplus \oplus \oplus\, s) = t$$
$$\vartheta(w_1, \oplus \oplus \oplus \neg p \mathcal{U} q) = t$$

# Lecture 6: Fix-point Logics

PLTL: linear time temporal logic

CTL: computation tree logic

PDL: propositional dynamic logic

LCK: logic of common knowledge

Look at CTL but using only one relation $R$ rather than $R = \sigma^*$