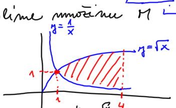
(1) $\iint_M xy \, dx\,dy$ $M:\ 1 \le x \le 4$, $\frac{1}{x} \le y \le \sqrt{x}$

A) nakreslíme množinu M



$\int_1^4 \left(\int_{\frac{1}{x}}^{\sqrt{x}} xy \, dy \right) dx = \int_1^4 x \left(\int_{\frac{1}{x}}^{\sqrt{x}} y \, dy \right) dx =$

$= \int_1^4 x \left[\frac{y^2}{2} \right]_{\frac{1}{x}}^{\sqrt{x}} dx =$

$= \int_1^4 x \left(\frac{x}{2} - \frac{\frac{1}{x^2}}{2} \right) dx =$

$= \int_1^4 x \left(\frac{x}{2} - \frac{1}{2x^2} \right) dx =$

$= \int_1^4 \frac{x^2}{2} - \frac{1}{2x} \, dx = \left[\frac{1}{2} \cdot \frac{x^3}{3} - \frac{1}{2} \ln|x| \right]_1^4$

$= \frac{1}{2} \cdot \frac{64}{3} - \frac{1}{2} \ln|4| - \left(\frac{1}{2} \cdot \frac{1}{3} - \frac{1}{2} \ln|1| \right) =$

$= \frac{32}{3} - \frac{1}{2} \ln 4 - \frac{1}{6} = \frac{64-1}{6} - \frac{1}{2} \cdot 2 \ln 2 =$

$= \frac{63}{6} - \ln 2 = \frac{21}{2} - \ln 2$

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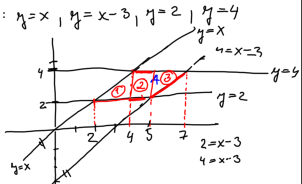
(2) $\iint_A f(x,y) \, dx\,dy$

A: $y=x$, $y=x-3$, $y=2$, $y=4$

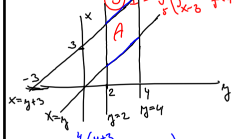


$\int_A \left(\int f(x,y) \, dy \right) dx = \iint_{(1)} f(x,y) \, dx\,dy + \iint_{(2)} f(x,y) \, dx\,dy + \iint_{(3)} f(x,y) \, dx\,dy =$

(1) $I = \int_2^4 \left(\int_2^x f(x,y) \, dy \right) dx$ (2) $I = \int_4^5 \left(\int_2^4 f(x,y) \, dy \right) dx$

(3) $I = \int_5^7 \left(\int_{x-3}^4 f(x,y) \, dy \right) dx$

$x-3=y$
$x=y+3$



$\int_2^4 \left(\int_y^{y+3} f(x,y) \, dx \right) dy$

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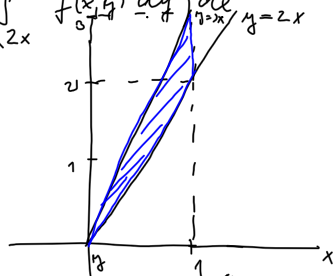
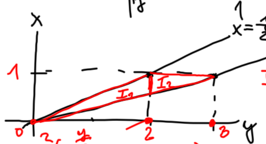
(3) ZAMĚŇTE POŘADÍ INTEGRACE

$\int_0^1 \left(\int_{2x}^{3x} f(x,y) \, dy \right) dx$





$I = I_1 + I_2$

$I_1 = \int_0^2 \left(\int_{\frac{y}{3}}^{\frac{y}{2}} f(x,y) \, dx \right) dy$ $I_2 = \int_2^3 \left(\int_{\frac{y}{3}}^{1} f(x,y) \, dx \right) dy$

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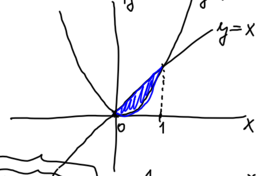
(4) Vypočtěte integrále

$\iint_A (x+y) \, dx\,dy$ $y=x^2$ $y=x$



$\int_0^1 \left(\int_{x^2}^x (x+y) \, dy \right) dx = \int_0^1 \left[xy + \frac{y^2}{2} \right]_{x^2}^x dx =$

$= \int_0^1 x^2 + \frac{x^2}{2} - x^3 - \frac{x^4}{2} \, dx =$

$= \int_0^1 \frac{3x^2}{2} - x^3 - \frac{x^4}{2} \, dx =$

$= \left[\frac{\cancel{3}}{2} \cdot \frac{x^3}{\cancel{3}} - \frac{x^4}{4} - \frac{1}{2} \cdot \frac{x^5}{5} \right]_0^1 = \frac{1}{2} - \frac{1}{4} - \frac{1}{10} =$

$= \frac{10-5-2}{20} = \frac{3}{20}$

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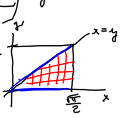
(5) $\int_0^{\sqrt{\frac{\pi}{2}}} \left(\int_y^{\sqrt{\frac{\pi}{2}}} y^2 \sin x^2 \, dx \right) dy =$

= VYMĚNÍME POŘADÍ



$= \int_0^{\sqrt{\frac{\pi}{2}}} \left(\int_0^x y^2 \sin x^2 \, dy \right) dx =$

$= \int_0^{\sqrt{\frac{\pi}{2}}} \sin x^2 \left[\frac{y^3}{3} \right]_0^x dx =$

$= \int_0^{\sqrt{\frac{\pi}{2}}} \sin x^2 \left(\frac{x^3}{3} - 0 \right) dx =$

$= \frac{1}{3} \int_0^{\sqrt{\frac{\pi}{2}}} x^3 \sin x^2 \, dx = \left| \begin{array}{l} t = x^2 \quad \left(\sqrt{\frac{\pi}{2}}\right)^2 = \frac{\pi}{2} \\ dt = 2x\,dx \\ dx = \frac{dt}{2x} \end{array} \right|$

$= \frac{1}{6} \int_0^{\frac{\pi}{2}} t \sin t \, dt = \overset{P.P.}{\left| \begin{array}{ll} u = t & u' = 1 \\ v' = \sin t & v = -\cos t \end{array} \right|}$

$= \frac{1}{6} \left([t \cdot (-\cos t)]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos t \, dt \right) =$

$= \frac{1}{6} \left([-t \cdot \cos t]_0^{\frac{\pi}{2}} + [\sin t]_0^{\frac{\pi}{2}} \right) =$

$= \frac{1}{6} (0 + 1 - 0) = \frac{1}{6}$

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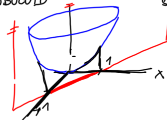
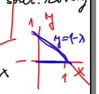
(6) Vypočtěte objem tělesa

$\underbrace{z = x^2+y^2}_{\text{PARABOLOID}}$, $\underbrace{x+y=1}_{\text{rovina}}$ ($y=-x+1$), $\underbrace{z=0,\ x=0,\ y=0}_{\text{souř. roviny}}$





$V = \underset{\text{VZOREC}}{\iiint_A 1 \, dx\,dy\,dz} = \int_0^1 \int_0^{1-x} \left(\int_0^{x^2+y^2} 1 \, dz \right) dy \, dx$

$= \int_0^1 \left(\int_0^{1-x} [z]_0^{x^2+y^2} dy \right) dx =$

$= \int_0^1 \left(\int_0^{1-x} x^2+y^2 \, dy \right) dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx =$

$\int_0^1 x^2(1-x) + \frac{(1-x)^3}{3} \, dx = \int_0^1 x^2 - x^3 + \frac{1-3x+3x^2-x^3}{3} \, dx$

$= \int_0^1 2x^2 - \frac{4}{3}x^3 - x + \frac{1}{3} \, dx =$

$= \left[\frac{2x^3}{3} - \frac{4}{3} \cdot \frac{x^4}{4} - \frac{x^2}{2} + \frac{x}{3} \right]_0^1 =$

$= \frac{2}{3} - \frac{1}{3} - \frac{1}{2} + \frac{1}{3} - 0 = \frac{1}{6}$

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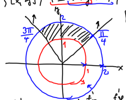
7) Spočítejte integrál

$$\iint_A 2(x^2+y^2)\,dA$$

$A = \{[x,y] \mid 1 \le x^2+y^2 \le 4,\ y \ge |x|\}$

$[x,y] = [\underbrace{r\cos\varphi}_{f}, \underbrace{r\sin\varphi}_{g}]$



Jakobián $|J| = \begin{vmatrix} f'_r & f'_\varphi \\ g'_r & g'_\varphi \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}$

$= r\cos^2\varphi + r\sin^2\varphi = r(\cos^2\varphi + \sin^2\varphi) = r$

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\int_1^2 r\cdot 2(r^2\cos^2\varphi + r^2\sin^2\varphi)\,dr\right)d\varphi =$$

POZOR! JAKOBIÁN!

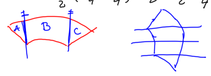
$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\int_1^2 2r^3\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}\,dr\right)d\varphi =$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\int_1^2 2r^3\,dr\right)d\varphi = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left[\cancel{2}\frac{r^4}{\cancel{4}_2}\right]_1^2 d\varphi =$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\frac{16}{2} - \frac{1}{2}\right)d\varphi = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{15}{2}\,d\varphi =$$

$$= \left[\frac{15}{2}\varphi\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{15}{2}\cdot\frac{3\pi}{4} - \frac{15}{2}\cdot\frac{\pi}{4} =$$

$$= \frac{15}{2}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \frac{15}{2}\cdot\frac{\pi}{2} = \frac{15\pi}{4}$$



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8) Spočítejte integrál

$$\int_0^1\left(\int_{-\sqrt{x-x^2}}^{\sqrt{1-x^2}} dy\right)dx$$

$y = \sqrt{1-x^2}\ /^2$ (NEEKV.) horní půlkružnice

$y^2 = 1-x^2 \Rightarrow x^2+y^2 = 1$

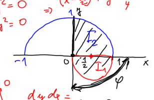
$y = -\sqrt{x-x^2}\ /^2$ předp. $y \le 0$

dolní půlkružnice

$y^2 = x - x^2$

$x^2 - x + y^2 = 0 \Rightarrow (x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$

$x^2 - 2\frac{x}{2} + y^2 = 0$



$$I_1 = \int_0^1 \int_{-\sqrt{x-x^2}}^{0} dy\,dx =$$

$y^2 \le x - x^2$

$r^2\sin^2\varphi \le r\cos\varphi - r^2\cos^2\varphi$

$r^2\underbrace{(\sin^2\varphi + \cos^2\varphi)}_{1} \le r\cos\varphi$

$r \le \cos\varphi$

$$I_1 = \int_{\frac{3\pi}{2}}^{2\pi}\left(\int_0^{\cos\varphi} r\,dr\right)d\varphi =$$

JAKOBIÁN

$$= \int_{\frac{3\pi}{2}}^{2\pi}\left[\frac{r^2}{2}\right]_0^{\cos\varphi} d\varphi = \int_{\frac{3\pi}{2}}^{2\pi}\frac{\cos^2\varphi}{2}\,d\varphi =$$

$$= \int_{\frac{3\pi}{2}}^{2\pi}\frac{1}{4}(1+\cos 2\varphi)\,d\varphi =$$

$\boxed{\begin{aligned} \cos 2\varphi &= \cos^2\varphi - \sin^2\varphi \\ &= \cos^2\varphi - (1-\cos^2\varphi) \\ &= 2\cos^2\varphi - 1 \\ \cos^2\varphi &= \frac{1+\cos 2\varphi}{2} \end{aligned}}$

$$= \frac{1}{4}\left[\varphi + \frac{(-\sin 2\varphi)}{2}\right]_{\frac{3\pi}{2}}^{2\pi} =$$

$$= \frac{1}{4}\left[2\pi - \left(\frac{3\pi}{2} + \frac{1}{2}\right)\right] = \frac{\pi}{2} - \frac{3\pi}{8} - \frac{1}{8} =$$

$$= \frac{\pi}{8} - \frac{1}{8}$$

$\boxed{I = \frac{\pi}{8} - \frac{1}{8} + \frac{\pi}{4}}$

$$I_2 = \int_0^1\left(\int_0^{\frac{\pi}{2}} r\,d\varphi\right)dr = \int_0^1 r\left(\int_0^{\frac{\pi}{2}} d\varphi\right)dr =$$

$$= \int_0^1 r\,[\varphi]_0^{\frac{\pi}{2}}\,dr = \int_0^1 \frac{\pi}{2}\,r\,dr = \left[\frac{\pi}{2}\cdot\frac{r^2}{2}\right]_0^1 = \frac{\pi}{4}$$

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