IA168 Algorithmic Game Theory

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Organization of This Course

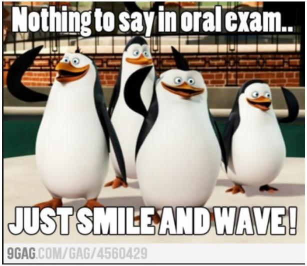
Sources:

- Lectures (slides, notes)
  - based on several sources
  - slides are prepared for lectures, some stuff on greenboard ($\Rightarrow$ attend the lectures)
- Books:
  - Nisan/Roughgarden/Tardos/Vazirani, **Algorithmic Game Theory**, Cambridge University, 2007.
    Available online for free:
    `http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf`
  - Tadelis, **Game Theory: An Introduction**, Princeton University Press, 2013

(I use various resources, so please, attend the lectures)

Evaluation

- **Oral exam**
- Homework (occasionally)



What is Algorithmic Game Theory?

First, what is the game theory?

According to the Oxford dictionary it is "the branch of mathematics concerned with the analysis of strategies for dealing with competitive situations where the outcome of a participant's choice of action depends critically on the actions of other participants"

According to Myerson it is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers"

What does the "algorithmic" mean?

- It means that we are "concerned with the computational questions that arise in game theory, and that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games."

Let's have a look at some examples ....

Prisoner's Dilemma

- Two suspects of a serious crime are arrested and imprisoned.
- Police has enough evidence of only petty theft, and to nail the suspects for the serious crime they need testimony from at least one of them.
- The suspects are interrogated separately without any possibility of communication.
- Each of the suspects is offered a deal: If he confesses (C) to the crime, he is free to go. The alternative is not to confess, that is remain silent (S).

Sentence depends on the behavior of both suspects.
The problem: What would the suspects do?

Prisoner's Dilemma – Solution(?)

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Rational "row" suspect (or his adviser) may reason as follows:

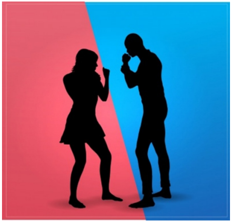
- If my colleague chooses C, then playing C gives me -5 and playing S gives -20.
- If my colleague chooses S, then playing C gives me 0 and playing S gives -1.

In both cases C is clearly better (it *strictly dominates* the other strategy). If the other suspect's reasoning is the same, both choose C and get 5 years sentence.

Where is the dilemma? There is a solution (S, S) which is better for both players but needs some "central" authority to control the players.

Are there always "dominant" strategies?

Nash equilibria – Battle of Sexes



- A couple agreed to meet this evening, but cannot recall if they will be attending the opera or a football match.
- The husband would like to go to the football game. The wife would like to go to the opera. Both would prefer to go to the same place rather than different ones.

If they cannot communicate, where should they go?

Nash equilibria – Battle of Sexes

Battle of Sexes can be modeled as a game of two players (Wife, Husband) with the following payoffs:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

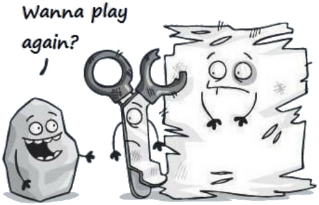
Apparently, no strategy of any player is dominant. A “solution”?

Note that whenever *both* players play O, then neither of them wants to *unilaterally* deviate from his strategy!

(O, O) is an example of a *Nash equilibrium* (as is (F, F))

Mixed Equilibria – Rock-Paper-Scissors

	R	P	S
R	$0,0$	$-1,1$	$1,-1$
P	$1,-1$	$0,0$	$-1,1$
S	$-1,1$	$1,-1$	$0,0$

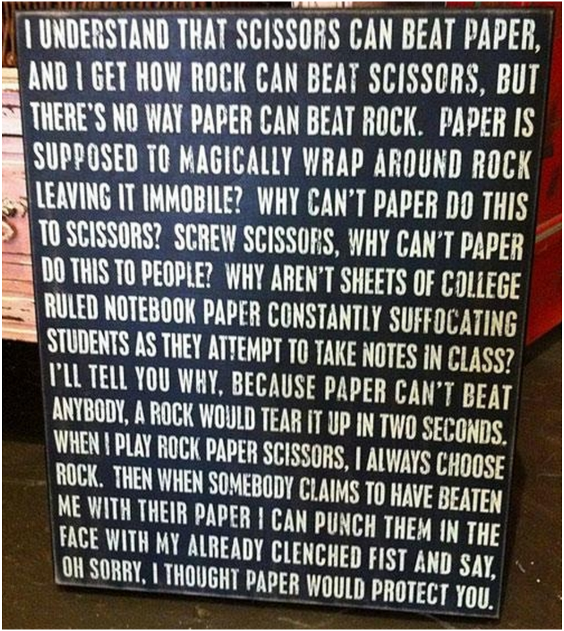


- This is an example of *zero-sum* games: whatever one of the players wins, the other one looses.
- What is an optimal behavior here? Is there a Nash equilibrium?

  Use *mixed strategies*: Each player plays each pure strategy with probability 1/3. The expected payoff of each player is 0 (even if one of the players changes his strategy, he still gets 0!).

How to algorithmically solve games in mixed strategies? (we shall use probability theory and linear programming)

Philosophical Issues in Games



Games of Incomplete Information

According to a study by the Institute of incomplete information 9 out of every 10.

In all previous games the players knew all details of the game they played, and this fact was a "common knowledge". This is not always the case.

Example: Sealed Bid Auction

- Two bidders are trying to purchase the same item.
- The bidders simultaneously submit bids b_1 and b_2 and the item is sold to the highest bidder at his bid price (first price auction)
- The payoff of the player 1 (and similarly for player 2) is calculated by

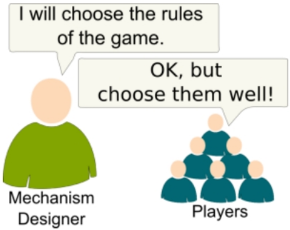
$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & b_1 > b_2 \\ \frac{1}{2}(v_1 - b_1) & b_1 = b_2 \\ 0 & b_1 < b_2 \end{cases}$$

  Here v_1 is the private value that player 1 assigns to the item and so the player 2 **does not know** u_1**.**

How to deal with such a game? Assume the "worst" private value? What if we have a partial knowledge about the private values?

Mechanism Design

Suppose you are the game designer. How would you design the game so that the “solutions” will satisfy certain “global objectives” ?



Examples:

- Sealed Bid Auctions: How would you design auction rules so that for every bidder, bidding the private value will be a dominant strategy?
- How would you design protocols (such as network protocols), to encourage “cooperation” (e.g., diminish congestion)?

This is an extremely hot topic of current research!

Inefficiency of Equilibria

In Prisoner's Dilemma, the selfish behavior of suspects (the Nash equilibrium) results in somewhat worse than ideal situation.

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

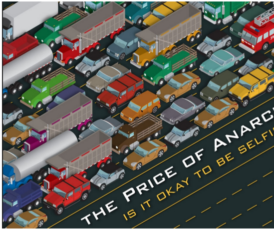
Defining a *welfare function* W which to every pair of strategies assigns the sum of payoffs, we get $W(C, C) = -10$ but $W(S, S) = -2$.

The ratio $\frac{W(C,C)}{W(S,S)} = 5$ measures the inefficiency of "selfish-behavior" (C, C) w.r.t. the optimal “centralized” solution.

Price of Anarchy is the maximum ratio between values of equilibria and the value of an optimal solution.

Inefficiency of Equilibria – Selfish Routing

Consider a transportation system where many agents are trying to get from some initial location to a destination. Consider the welfare to be the average time for an agent to reach the destination. There are two versions:



- “Centralized”: A central authority tells each agent where to go.
- “Decentralized”: Each agent selfishly minimizes his travel time.

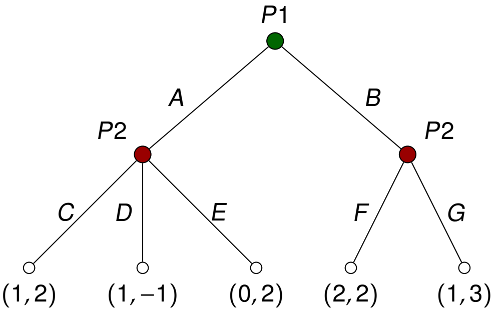
Price of Anarchy measure the ratio between average travel time in these two cases.

Problem: Bound the price of anarchy over all routing games?

Dynamic Games

So far we have seen games in *strategic form* that are unable to capture games that unfold over time (such as chess).

For such purpose we need to use *extensive form* games:



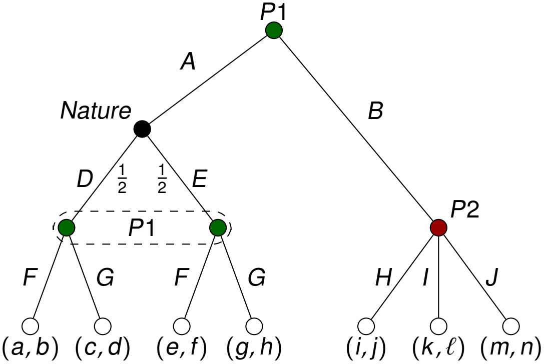
How to "solve" such games?

What is their relationship to the strategic form games?

Chance and Imperfect Information

Some decisions in the game tree may be by chance and controlled by neither player (e.g. Poker, Backgammon, etc.)

Sometimes a player may not be able to distinguish between several "positions" because he does not know all the information in them (Think a card game with opponent's cards hidden).



Again, how to solve such games?

Games in Computer Science

Game theory is a core foundation of mathematical economics. But what does it have to do with CS?

- Games in AI: modeling of “rational” agents and their interactions.
- Games in Algorithms: several game theoretic problems have a very interesting algorithmic status and are solved by interesting algorithms
- Games in modeling and analysis of reactive systems: program inputs viewed “adversarially”, bisimulation games, etc.
- Games in computational complexity: Many complexity classes are definable in terms of games: PSPACE, polynomial hierarchy, etc.
- Games in Logic: modal and temporal logics, Ehrenfeucht-Fraisse games, etc.

Games in Computer Science

Games, the Internet and E-commerce: An extremely active research area at the intersection of CS and Economics

Basic idea: “The internet is a HUGE experiment in interaction between agents (both human and automated)”

How do we set up the rules of this game to harness “socially optimal” results?

Summary and Brief Overview

This is a *theoretical* course aimed at some fundamental results of game theory, often related to computer science

- We start with strategic form games (such as the Prisoner's dilemma), investigate several solution concepts (dominance, equilibria) and related algorithms (in particular, Lemke-Howson algorithm for computing Nash Eq.)
- Subsequently, we move on to incomplete information games, auctions, and mechanism design
- Then consider (in)efficiency of equilibria (such as the Price of Anarchy) and its properties on important classes of routing and network formation games.
- Then we consider repeated games which allow players to learn from history and/or to react to deviations of the other players.
- Remaining time will be devoted to selected topics from extensive form games, games on graphs etc.

Static Games of Complete Information

Strategic-Form Games

Solution concepts

Static Games of Complete Information – Intuition

Proceed in two steps:

1. Each player *simultaneously and independently* chooses a *strategy*. This means that players play without observing strategies chosen by other players.
2. Conditional on the players' strategies, *payoffs* are distributed to all players.

Complete information means that the following is *common knowledge* among players:

- all possible strategies of all players,
- what payoff is assigned to each combination of strategies.

Definition 1

A fact E is a *common knowledge* among players $\{1, \ldots, n\}$ if for every sequence $i_1, \ldots, i_k \in \{1, \ldots, n\}$ we have that i_1 knows that i_2 knows that ... i_{k-1} knows that i_k knows E.

The goal of each player is to maximize his payoff (and this fact is common knowledge).

Strategic-Form Games

To formally represent static games of complete information we define *strategic-form games*.

Definition 2

A game in *strategic-form* (or normal-form) is an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, in which:

- $N = \{1, 2, \ldots, n\}$ is a finite set of *players*.
- S_i is a set of *(pure) strategies* of player i, for every $i \in N$.

  A *strategy profile* is a vector of strategies of all players $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$.

  We denote the set of all strategy profiles by $S = S_1 \times \cdots \times S_n$.
- $u_i : S \to \mathbb{R}$ is a function associating each strategy profile $s = (s_1, \ldots, s_n) \in S$ with the *payoff* $u_i(s)$ to player i, for every player $i \in N$.

Definition 3

A *zero-sum* game G is one in which for all $s = (s_1, \ldots, s_n) \in S$ we have $u_1(s) + u_2(s) + \cdots + u_n(s) = 0$.

Example: Prisoner's Dilemma

- $N = \{1, 2\}$
- $S_1 = S_2 = \{S, C\}$
- u_1, u_2 are defined as follows:
  - $u_1(C, C) = -5$, $u_1(C, S) = 0$, $u_1(S, C) = -20$, $u_1(S, S) = -1$
  - $u_2(C, C) = -5$, $u_2(C, S) = -20$, $u_2(S, C) = 0$, $u_2(S, S) = -1$

(Is it zero sum?)

We usually write payoffs in the following form:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

or as two matrices:

	C	S
C	-5	0
S	-20	-1

	C	S
C	-5	-20
S	0	-1

Example: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good. Denote by q_1 and q_2 quantities produced by firms 1 and 2, resp.
- The total quantity of products in the market is $q_1 + q_2$.
- The price of each item is $\kappa - q_1 - q_2$ (here κ is a positive constant)
- Firms 1 and 2 have per item production costs c_1 and c_2, resp.

Question: How these firms are going to behave?

We may model the situation using a strategic-form game.

Strategic-form game model $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1$
  $u_2(q_1, q_2) = q_2(\kappa - q_1 - q_2) - q_2 c_2$

Solution Concepts

A *solution concept* is a method of analyzing games with the objective of restricting the set of *all possible outcomes* to those that are *more reasonable than others.*

We will use term *equilibrium* for any one of the strategy profiles that emerges as one of the solution concepts' predictions.
(I follow the approach of Steven Tadelis here, it is not completely standard)

Example 4

Nash equilibrium is a solution concept. That is, we "solve" games by finding Nash equilibria and declare them to be reasonable outcomes.

Assumptions

Throughout the lecture we assume that:

1. Players are **rational**: a *rational* player is one who chooses his strategy to maximize his payoff.
2. Players are **intelligent**: An *intelligent* player knows everything about the game (actions and payoffs) and can make any inferences about the situation that we can make
3. **Common knowledge**: The fact that players are rational and intelligent is a common knowledge among them.
4. **Self-enforcement**: Any prediction (or equilibrium) of a solution concept must be *self-enforcing*.

Here 4. implies non-cooperative game theory: Each player is in control of his actions, and he will stick to an action only if he finds it to be in his best interest.

Evaluating Solution Concepts

In order to evaluate our theory as a methodological tool we use the following criteria:

1. Existence (i.e. How often does it apply?): Solution concept should apply to a wide variety of games.

   E.g. We prove that mixed Nash equilibria exist in all two player finite strategic-form games.

2. Uniqueness (How much does it restrict behavior?): We demand our solution concept to restrict the behavior as much as possible.

   E.g. So called strictly dominant strategy equilibria are always unique as opposed to Nash eq.

The basic notion for evaluating "social outcome" is the following

Definition 5

A strategy profile $s \in S$ *Pareto dominates* a strategy profile $s' \in S$ if $u_i(s) \geq u_i(s')$ for all $i \in N$, and $u_i(s) > u_i(s')$ for at least one $i \in N$.
A strategy profile $s \in S$ is *Pareto optimal* if it is not Pareto dominated by any other strategy profile.

We will see more measures of social outcome later.

Solution Concepts – Pure Strategies

We will consider the following solution concepts:

- strict dominant strategy equilibrium
- iterated elimination of strictly dominated strategies (IESDS)
- rationalizability
- Nash equilibria

For now, let us concentrate on

pure strategies only!

I.e., no mixed strategies are allowed. We will generalize to mixed setting later.

Notation

- Let $N = \{1, \ldots, n\}$ be a finite set and for each $i \in N$ let X_i be a set. Let $X := \prod_{i \in N} X_i = \{(x_1, \ldots, x_n) \mid x_j \in X_j, j \in N\}$.
  - For $i \in N$ we define $X_{-i} := \prod_{j \neq i} X_j$, i.e.,

    $$X_{-i} = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \mid x_j \in X_j, \forall j \neq i\}$$

  - An element of X_{-i} will be denoted by

    $$x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

    We slightly abuse notation and write (x_i, x_{-i}) to denote $(x_1, \ldots, x_i, \ldots, x_n) \in X$.

Strict Dominance in Pure Strategies

Definition 6

Let $s_i, s_i' \in S_i$ be strategies of player i. Then s_i' is *strictly dominated* by s_i (write $s_i \succ s_i'$) if for any possible combination of the other players' strategies, $s_{-i} \in S_{-i}$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

Claim 1

An intelligent and rational player will never play a strictly dominated strategy.

Clearly, intelligence implies that the player should recognize dominated strategies, rationality implies that the player will avoid playing them.

Strictly Dominant Strategy Equilibrium in Pure Str.

Definition 7

$s_i \in S_i$ is *strictly dominant* if every other pure strategy of player i is strictly dominated by s_i.

Observe that every player has at most one strictly dominant strategy, and that strictly dominant strategies do not have to exist.

Claim 2

Any rational player will play the strictly dominant strategy (if it exists).

Definition 8

A strategy profile $s \in S$ is a *strictly dominant strategy equilibrium* if $s_i \in S_i$ is strictly dominant for all $i \in N$.

Corollary 9

If the strictly dominant strategy equilibrium exists, it is unique and rational players will play it.

Is the strictly dominant strategy equilibrium always Pareto optimal?

Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the strictly dominant strategy equilibrium (the only profile that is not Pareto optimal!).

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

no strictly dominant strategies exist.

Indiana Jones and the Last Crusade

(Taken from Dixit & Nalebuff's "The Art of Strategy" and a lecture of Robert Marks)

Indiana Jones, his father, and the Nazis have all converged at the site of the Holy Grail. The two Joneses refuse to help the Nazis reach the last step. So the Nazis shoot Indiana's dad. Only the healing power of the Holy Grail can save the senior Dr. Jones from his mortal wound. Suitably motivated, Indiana leads the way to the Holy Grail. But there is one final challenge. He must choose between literally scores of chalices, only one of which is the cup of Christ. While the right cup brings eternal life, the wrong choice is fatal. The Nazi leader impatiently chooses a beautiful gold chalice, drinks the holy water, and dies from the sudden death that follows from the wrong choice. Indiana picks a wooden chalice, the cup of a carpenter. Exclaiming "There's only one way to find out" he dips the chalice into the font and drinks what he hopes is the cup of life. Upon discovering that he has chosen wisely, Indiana brings the cup to his father and the water heals the mortal wound.

Indiana Jones and the Last Crusade (cont.)

Indy Goofed

- Although this scene adds excitement, it is somewhat embarrassing that such a distinguished professor as Dr. Indiana Jones would overlook his dominant strategy.
- He should have given the water to his father without testing it first.
  - If Indiana has chosen the right cup, his father is still saved.
  - If Indiana has chosen the wrong cup, then his father dies but Indiana is spared.
- Testing the cup before giving it to his father doesn't help, since if Indiana has made the wrong choice, there is no second chance – Indiana dies from the water and his father dies from the wound.