# Strategic-Form Games – Conclusion

We have considered *static games of complete information*, i.e., "one-shot" games where the players know exactly what game they are playing.
We modeled such games using *strategic-form games*.

We have considered both pure strategy setting and mixed strategy setting.

In both cases, we considered four solution concepts:

- Strictly dominant strategies
- Iterative elimination of strictly dominated strategies
- Rationalizability (i.e., iterative elimination of strategies that are never best responses)
- Nash equilibria

# Strategic-Form Games – Conclusion

In pure strategy setting:

1. Strictly dominant strategy equilibrium survives IESDS, rationalizability and is the unique Nash equilibrium (if it exists)
2. In finite games, rationalizable equilibria survive IESDS, IESDS preserves the set of Nash equilibria
3. In finite games, rationalizability preserves Nash equilibria

In mixed setting:

1. In finite two player games, IESDS and rationalizability coincide.
2. Strictly dominant strategy equilibrium survives IESDS (rationalizability) and is the unique Nash equilibrium (if it exists)
3. In finite games, IESDS (rationalizability) preserves Nash equilibria

The proofs for 2. and 3. in the mixed setting are similar to corresponding proofs in the pure setting.

# Algorithms

- Strictly dominant strategy equilibria coincide in pure and mixed settings, and can be computed in polynomial time.
- IESDS and rationalizability can be implemented in polynomial time in the pure setting as well as in the mixed setting

  In the mixed setting, linear programming is needed to implement one step of IESDS (rationalizability).
- Nash equilibria can be computed for two-player games
  - in polynomial time for zero-sum games
    (using von Neumann's theorem and linear programming)
  - in exponential time using support enumeration
  - in PPAD using Lemke-Howson

# Complexity of Nash Eq. – FNP (Roughly)

Let $R$ be a binary relation on words (over some alphabet) that is polynomial-time computable and polynomially balanced.

I.e., membership to $R$ is decidable in polynomial time, and $(x, y) \in R$ implies $|y| \leq |x|^k$ where $k$ is independent of $x, y$.

A *search problem* associated with $R$ is this: Given an input $x$, return a $y$ such that $(x, y) \in R$ if such $y$ exists, and return "NO" otherwise.

Note that the problem of computing NE can be seen as a search problem $R$ where $(x, y) \in R$ means that $x$ is a strategic-form game and $y$ is a Nash equilibrium of polynomial size. (We already know from support enumeration that there is a NE of polynomial size.)

The class of all search problems is called FNP. A class FP $\subseteq$ FNP contains all search problem that can be solved in polynomial time.

A search problem determined by $R$ is *polynomially reducible* to a search problem $R'$ iff there exist polynomially computable functions $f, g$ such that

- if $(x, y) \in R$ for some $y$, then $(f(x), y') \in R'$ for some $y'$
- if $(f(x), y) \in R'$, then $(x, g(y)) \in R$
- if $(f(x), y) \notin R'$ for all $y$, then $(x, y) \notin R$ for all $y$

# Complexity of Nash Eq. – PPAD (Roughly)

The class PPAD is defined by specifying one of its complete problems (w.r.t. the polynomial time reduction) known as *End-Of-The-Line*:

- **Input:** Two *Boolean circuits (with basis $\wedge, \vee, \neg$)* $S$ and $P$, each with $m$ input bits and $m$ output bits, such that $P(0^m) = 0^m \neq S(0^m)$.
- **Problem:** Find an input $x \in \{0,1\}^m$ such that $P(S(x)) \neq x$ or $S(P(x)) \neq x \neq 0^m$.

Intuition: *End-Of-The-Line* creates a directed graph $H_{S,P}$ with vertex set $\{0,1\}^m$ and an edge from $x$ to $y$ whenever both $y = S(x)$ ("successor") and $x = P(y)$ ("predecessor").

All vertices of $H_{S,P}$ have indegree and outdegree at most one. There is at least one source (i.e., $x$ satisfying $P(x) = x$, namely $0^m$), so there is at least one sink (i.e., $x$ satisfying $S(x) = x$).

The goal is to find either a source or a sink different from $0^m$.

**Theorem 53**

*The problem of computing Nash equilibria is complete for PPAD.*
*That is, Nash belongs to PPAD and End-Of-The-Line is polynomially reducible to Nash.*

# Loose Ends – Modes of Dominance

Let $\sigma_i, \sigma'_i \in \Sigma_i$. Then $\sigma'_i$ is *strictly dominated* by $\sigma_i$ if
$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma'_i, \sigma_{-i})$ for all $\sigma_{-i} \in \Sigma_{-i}$.

Let $\sigma_i, \sigma'_i \in \Sigma_i$. Then $\sigma'_i$ is *weakly dominated* by $\sigma_i$ if
$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$ for all $\sigma_{-i} \in \Sigma_{-i}$ and there is $\sigma'_{-i} \in \Sigma_{-i}$ such that $u_i(\sigma_i, \sigma'_{-i}) > u_i(\sigma'_i, \sigma'_{-i})$.

Let $\sigma_i, \sigma'_i \in \Sigma_i$. Then $\sigma'_i$ is *very weakly dominated* by $\sigma_i$ if
$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$ for all $\sigma_{-i} \in \Sigma_{-i}$.

A strategy is (strictly, weakly, very weakly) dominant in mixed strategies if it (strictly, weakly, very weakly) dominates any other mixed strategy.

**Claim 4**

*Any mixed strategy profile $\sigma \in \Sigma$ such that each $\sigma_i$ is very weakly dominant in mixed strategies is a mixed Nash equilibrium.*

The same claim can be proved in pure strategy setting.

# Dynamic Games of Complete Information

## Extensive-Form Games

## Definition

## Sub-Game Perfect Equilibria

# Dynamic Games of Perfect Information (Motivation)

Static games (modeled using strategic-form games) cannot capture games that unfold over time.

In particular, as all players move simultaneously, there is no way how to model situations in which order of moves is important.

Imagine e.g. chess where players take turns, in every round a player knows all turns of the opponent before making his own turn.

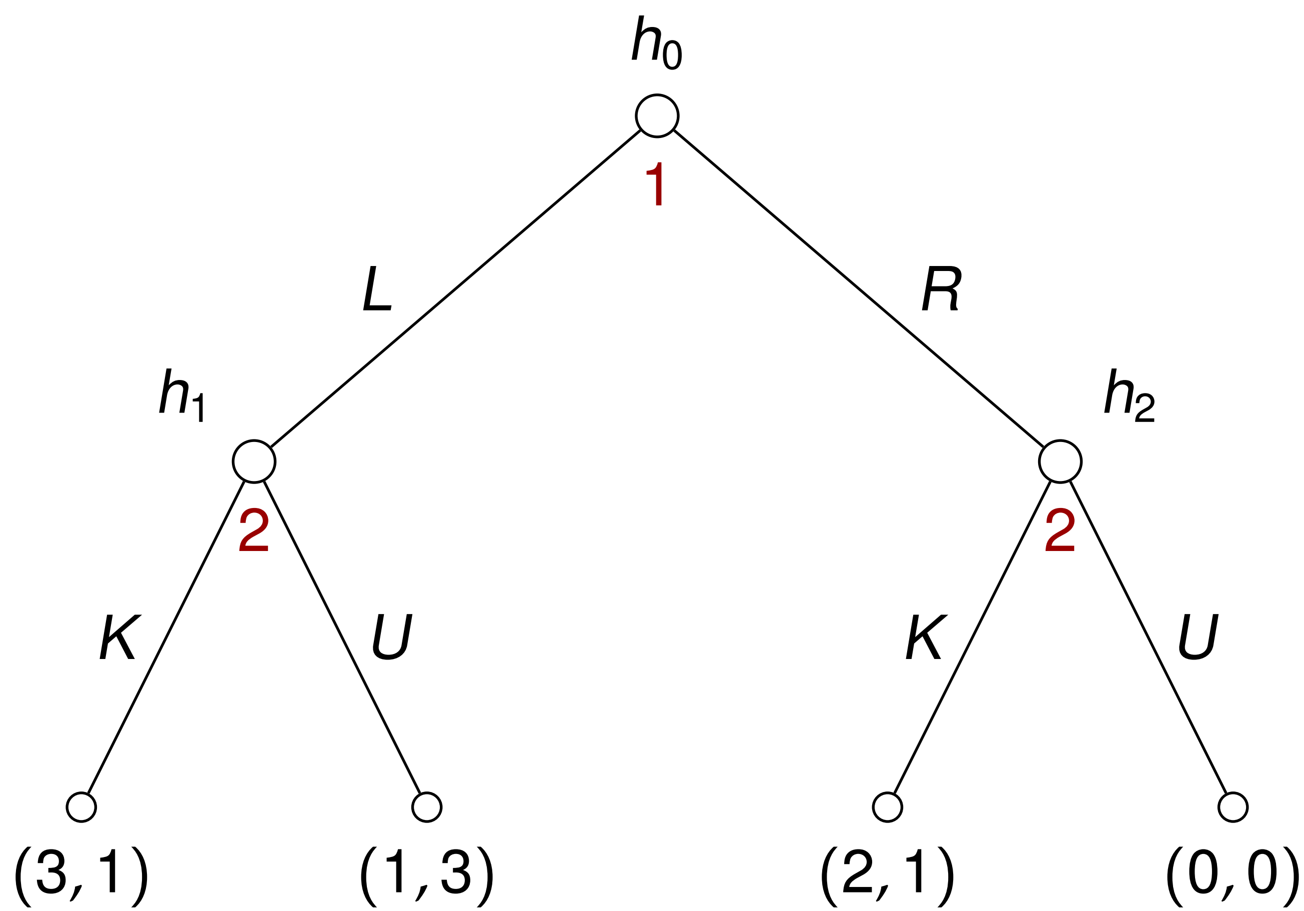
There are many examples of dynamic games: markets that change over time, political negotiations, models of computer systems, etc.

We model dynamic games using *extensive-form games*, a tree like model that allows to express sequential nature of games.

We start with perfect information games, where each player always knows results of all previous moves.

Then generalize to imperfect information, where players may have only partial knowledge of these results (e.g. most card games).

# Perfect-Info. Extensive-Form Games (Example)




Here $h_0, h_1, h_2$ are non-terminal nodes, leaves are terminal nodes.

Each non-terminal node is owned by a player who chooses an action.

E.g. $h_1$ is owned by player 2 who chooses either $K$ or $U$

Every action results in a transition to a new node.

Choosing $L$ in $h_0$ results in a move to $h_1$

When a play reaches a terminal node, players collect payoffs.

E.g. the left most terminal node gives 3 to player 1 and 1 to player 2.

# Perfect-Information Extensive-Form Games

A *perfect-information extensive-form game* is a tuple $G = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$ where

- $N = \{1, \ldots, n\}$ is a set of $n$ *players*, $A$ is a (single) set of *actions*,
- $H$ is a set of *non-terminal* (choice) nodes, $Z$ is a set of *terminal* nodes (assume $Z \cap H = \emptyset$), denote $\mathcal{H} = H \cup Z$,
- $\chi : H \rightarrow \left(2^A \setminus \{\emptyset\}\right)$ is the *action function*, which assigns to each choice node a *non-empty* set of *enabled* actions,
- $\rho : H \rightarrow N$ is the *player function*, which assigns to each non-terminal node a player $i \in N$ who chooses an action there, we define $H_i := \{h \in H \mid \rho(h) = i\}$,
- $\pi : H \times A \rightarrow \mathcal{H}$ is the *successor function*, which maps a non-terminal node and an action to a new node, such that
  - $h_0$ is the only node that is not in the image of $\pi$ (the root)
  - for all $h_1, h_2 \in H$ and for all $a_1 \in \chi(h_1)$ and all $a_2 \in \chi(h_2)$, if $\pi(h_1, a_1) = \pi(h_2, a_2)$, then $h_1 = h_2$ and $a_1 = a_2$,
- $u = (u_1, \ldots, u_n)$, where each $u_i : Z \rightarrow \mathbb{R}$ is a *payoff function* for player $i$ in the terminal nodes of $Z$.

# Some Notation

A *path* from $h \in \mathcal{H}$ to $h' \in \mathcal{H}$ is a sequence $h_1 a_2 h_2 a_3 h_3 \cdots h_{k-1} a_k h_k$ where $h_1 = h$, $h_k = h'$ and $\pi(h_{j-1}, a_j) = h_j$ for every $1 < j \leq k$.
Note that, in particular, $h$ is a path from $h$ to $h$.

**Assumption:** For every $h \in \mathcal{H}$ there is a unique path from $h_0$ to $h$ and there is no infinite path (i.e., a sequence $h_1 a_2 h_2 a_3 h_3 \cdots$ such that $\pi(h_{j-1}, a_j) = h_j$ for every $j > 1$).

Note that the assumption is satisfied when $\mathcal{H}$ is finite.
Indeed, uniqueness follows immediately from the definition of $\pi$. Now let $X$ be the set of all $h'$ from which there is a path to $h$. If $h_0 \in X$ we are done. Otherwise, let $h'$ be a node of $X$ with the longest path to $h$. As $h' \neq h_0$, there is $h''$ and $a \in \chi(h'')$ such that $h' = \pi(h'', a)$. But then there is a path from $h''$ to $h$ that is longer than the path from $h'$, a contradiction.

The above claim implies that every perfect-information extensive-form game can be seen as a game on a *rooted tree* $(\mathcal{H}, E, h_0)$ where

- $H \cup Z$ is a set of nodes,
- $E \subseteq \mathcal{H} \times \mathcal{H}$ is a set of edges defined by $(h, h') \in E$ iff $h \in H$ and there is $a \in \chi(h)$ such that $\pi(h, a) = h'$,
- $h_0$ is the root.

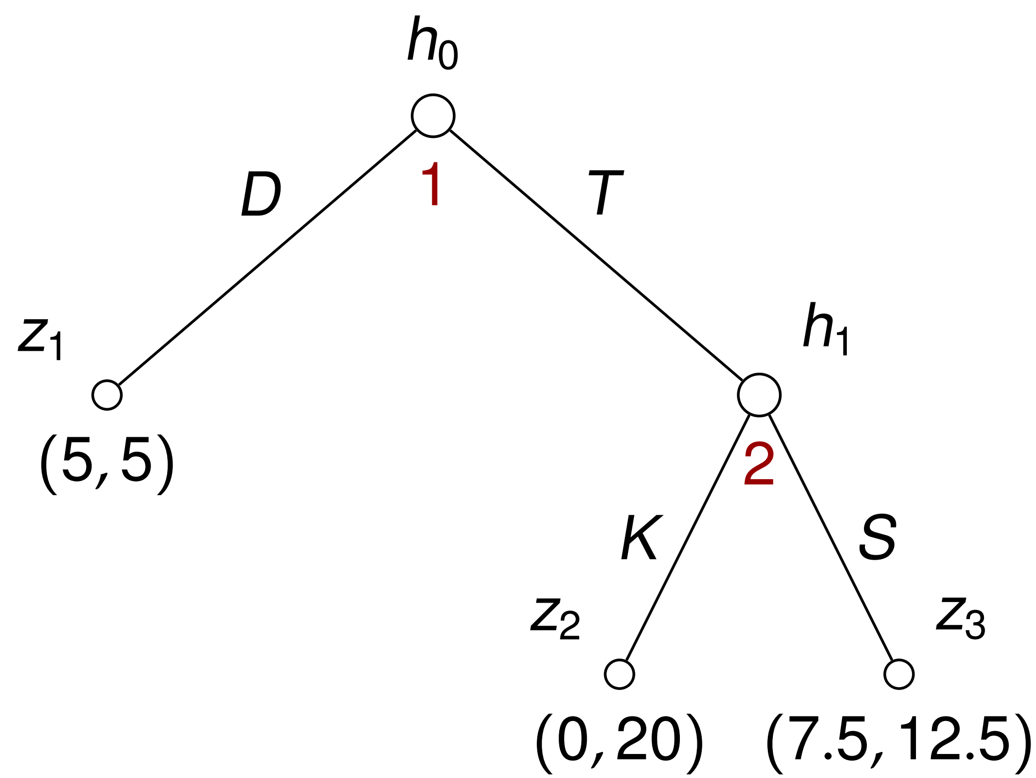
# Some More Notation

$h'$ is a *child* of $h$, and $h$ is a *parent* of $h'$ if there is $a \in \chi(h)$ such that $h' = \pi(h, a)$.

$h' \in \mathcal{H}$ is *reachable* from $h \in \mathcal{H}$ if there is a path from $h$ to $h'$.

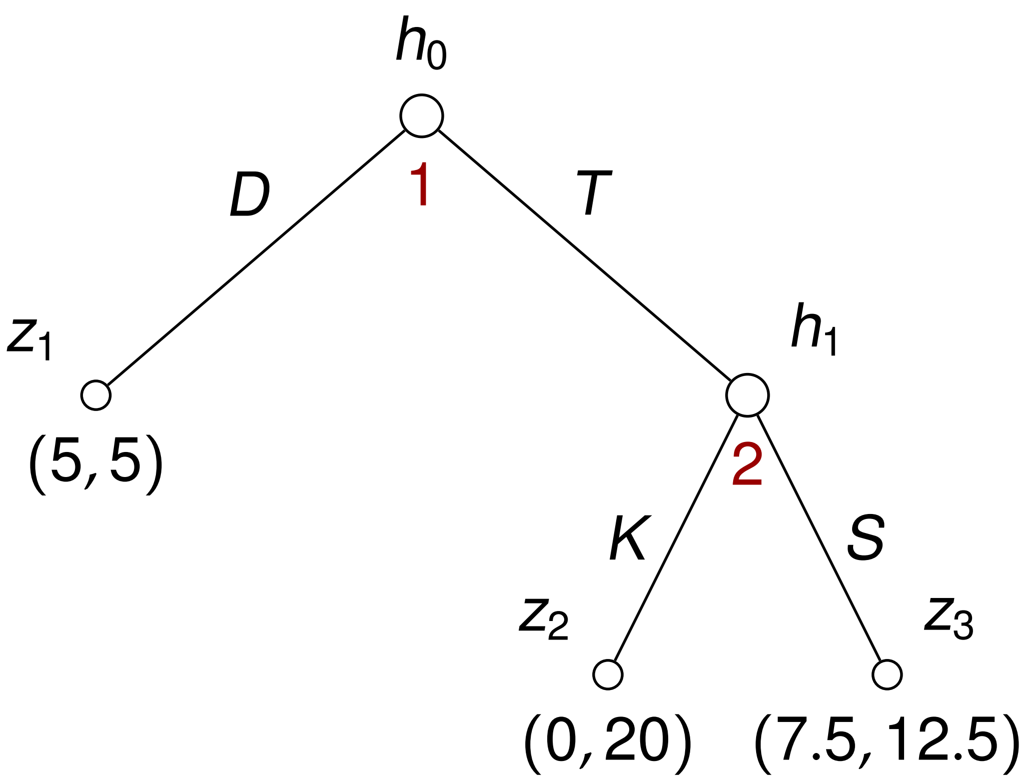
If $h'$ is reachable from $h$ we say that $h'$ is a descendant of $h$ and $h$ is an ancestor of $h'$ (note that, by definition, $h$ is both a descendant and an ancestor of itself).

# Example: Trust Game




- Two players, both start with 5$
- Player 1 either distrusts (D) player 2 and keeps the money (payoffs $(5, 5)$), or trusts (T) player 2 and passes 5$ to player 2
- If player 1 chooses to trust player 2, the money is tripled by the experimenter and sent to player 2.
- Player 2 may either keep (K) the additional 15$ (resulting in $(0, 20)$), or share (S) it with player 1 (resulting in $(7.5, 12.5)$)

# Example: Trust Game (Cont.)




- $N = \{1, 2\}$, $A = \{D, T, K, S\}$
- $H = \{h_0, h_1\}$, $Z = \{z_1, z_2, z_3\}$
- $\chi(h_0) = \{D, T\}$, $\chi(h_1) = \{K, S\}$
- $\rho(h_0) = 1$, $\rho(h_1) = 2$
- $\pi(h_0, D) = z_1$, $\pi(h_0, T) = h_1$, $\pi(h_1, K) = z_2$, $\pi(h_1, S) = z_3$
- $u_1(z_1) = 5$, $u_1(z_2) = 0$, $u_1(z_3) = 7.5$, $u_2(z_1) = 5$, $u_2(z_2) = 20$, $u_2(z_3) = 12.5$

# Stackelberg Competition

Very similar to Cournot duopoly ...

- Two identical firms, players 1 and 2, produce some good.
  Denote by $q_1$ and $q_2$ quantities produced by firms 1 and 2, resp.
- The total quantity of products in the market is $q_1 + q_2$.
- The price of each item is $\kappa - q_1 - q_2$ where $\kappa > 0$ is fixed.
- Firms have a common per item production cost $c$.

Except that ...

- As opposed to Cournot duopoly, the firm 1 moves first, and chooses the quantity $q_1 \in [0, \infty)$.
- Afterwards, the firm 2 chooses $q_2 \in [0, \infty)$ (knowing $q_1$) and then the firms get their payoffs.

# Stackelberg Competition – Extensive-Form Model

An extensive-form game model:

- $N = \{1, 2\}$
- $A = [0, \infty)$
- $H = \{h_0, h_1^{q_1} \mid q_1 \in [0, \infty)\}$
- $Z = \{z^{q_1, q_2} \mid q_1, q_2 \in [0, \infty)$
- $\chi(h_0) = [0, \infty), \qquad \chi(h_1^{q_1}) = [0, \infty)$
- $\rho(h_0) = 1, \qquad \rho(h_1^{q_1}) = 2$
- $\pi(h_0, q_1) = h_1^{q_1}, \qquad \pi(h_1^{q_1}, q_2) = z^{q_1, q_2}$
- The payoffs are
  - $u_1(z^{q_1, q_2}) = q_1(\kappa - q_1 - q_2) - q_1 c$
  - $u_2(z^{q_1, q_2}) = q_2(\kappa - q_1 - q_2) - q_2 c$

# Example: Chess (a bit simplified)

There are infinitely many representations of chess, this one is different from the one presented at the lecture.

- $N = \{1, 2\}$
- Denoting *Boards* the set of all (appropriately encoded) board positions, we define $\mathcal{H} = B \times \{1, 2\}$ where

  $$B = \{w \in Boards^+ \mid \text{ no board repeats } \geq 3 \text{ times in } w\}$$

  (Here $Boards^+$ is the set of all non-empty sequences of boards)
- $Z$ consists of all nodes $(wb, i)$ (here $b \in Boards$) where either $b$ is checkmate for player $i$, or $i$ does not have a move in $b$, or every move of $i$ in $b$ leads to a board with two occurrences in $w$
- $\chi(wb, i)$ is the set of all legal moves of player $i$ in $b$
- $\rho(wb, i) = i$
- $\pi$ is defined by $\pi((wb, i), a) = (wbb', 2 - i + 1)$ where $b'$ is obtained from $b$ according to the move $a$
- $h_0 = (b_0, 1)$ where $b_0$ is the initial board
- $u_j(wb, i) \in \{1, 0, -1\}$, here 1 means "win", 0 means "draw", and $-1$ means "loss" for player $j$

# Pure Strategies

Let $G = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$ be a perfect-information extensive-form game.

**Definition 54**
A *pure strategy* of player $i$ in $G$ is a function $s_i : H_i \to A$ such that for every $h \in H_i$ we have that $s_i(h) \in \chi(h)$.

We denote by $S_i$ the set of all pure strategies of player $i$ in $G$.

Denote by $S = S_1 \times \cdots \times S_n$ the set of all pure strategy profiles.

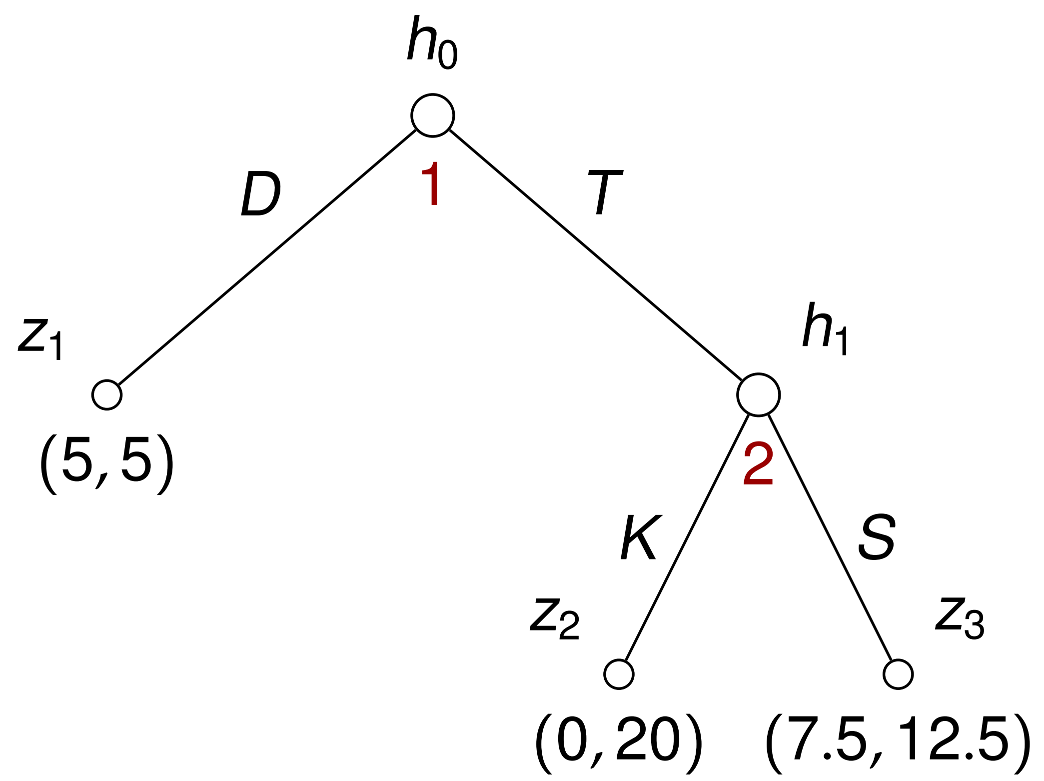
Note that each pure strategy profile $s \in S$ determines a unique path $w_s = h_0 a_1 h_1 \cdots h_{k-1} a_k h_k$ from $h_0$ to a terminal node $h_k$ by

$$a_j = s_{\rho(h_{j-1})}(h_{j-1}) \qquad \forall 0 < j \leq k$$

Denote by $O(s)$ the terminal node reached by $w_s$.

Abusing notation a bit, we denote by $u_i(s)$ the value $u_i(O(s))$ of the payoff for player $i$ when the terminal node $O(s)$ is reached using strategies of $s$.

# Example: Trust Game




A pure strategy profile $(s_1, s_2)$ where

$$s_1(h_0) = T \qquad \text{and} \qquad s_2(h_1) = K$$

is usually written as *TK* (BFS & left to right traversal) determines the path $h_0 \, T \, h_1 \, K \, z_2$

The resulting payoffs: $u_1(s_1, s_2) = 0$ and $u_2(s_1, s_2) = 20$.

# Extensive-Form vs Strategic-Form

The extensive-form game $G$ determines the *corresponding strategic-form game* $\bar{G} = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

Here note that the set of players $N$ and the sets of pure strategies $S_i$ are the same in $G$ and in the corresponding game.

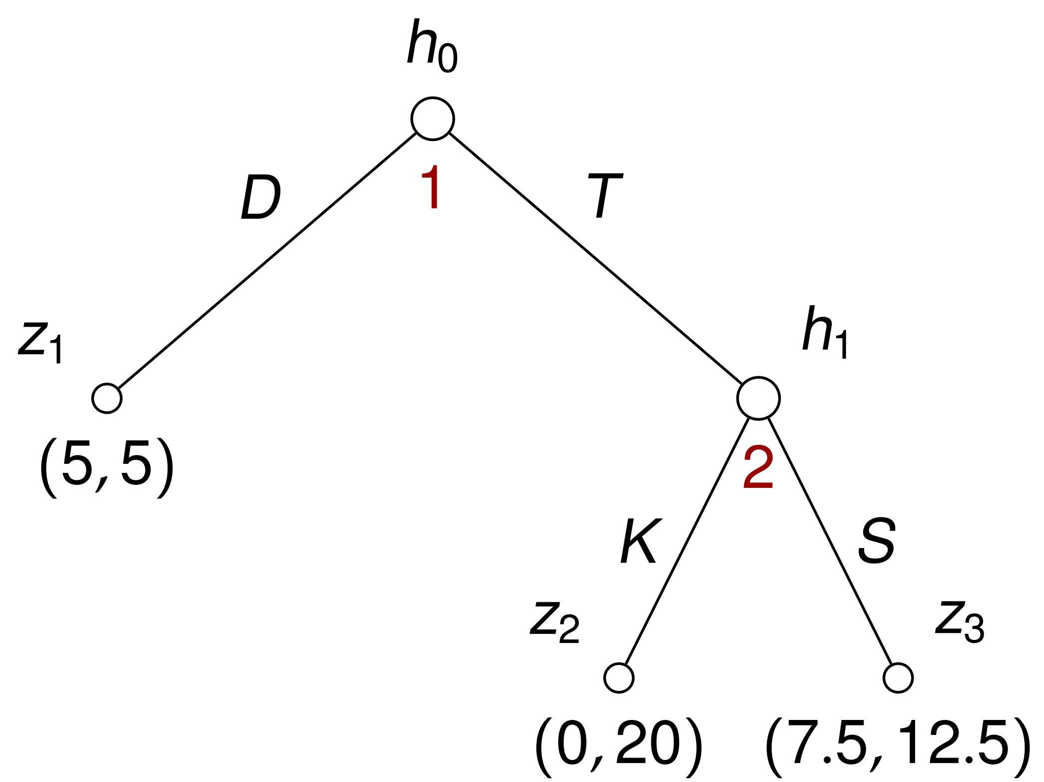
The payoff functions $u_i$ in $\bar{G}$ are understood as functions on the pure strategy profiles of $S = S_1 \times \cdots \times S_n$.

With this definition, we may apply all solution concepts and algorithms developed for strategic-form games to the extensive form games.
We often consider the extensive-form to be only a different way of representing the corresponding strategic-form game and do not distuinguish between them.

There are some issues, namely whether all notions from strategic-form area make sense in the extensive-form. Also, naive application of algorithms may result in unnecessarily high complexity.

For now, let us consider pure strategies only!

# Example: Trust Game

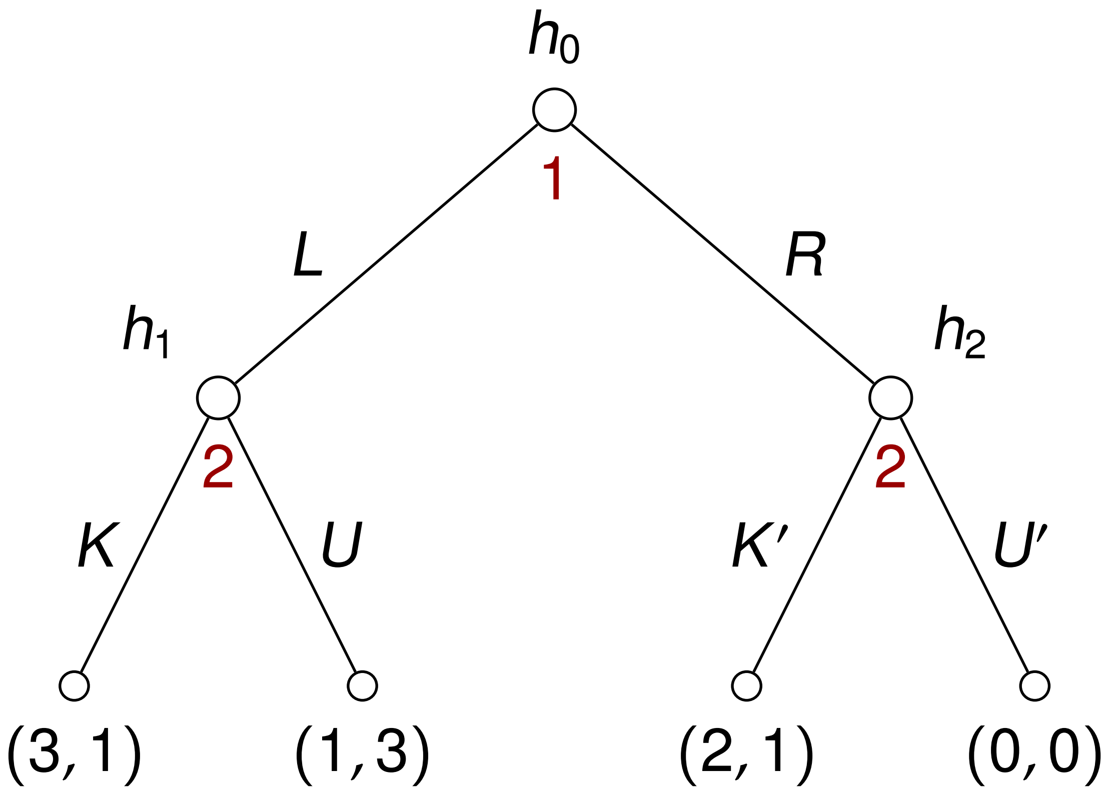



Is any strategy strictly (weakly, very weakly) dominant?

Is any strategy never best response?

Is there a Nash equilibrium in pure strategies ?

# Example




Find all pure strategies of both players.

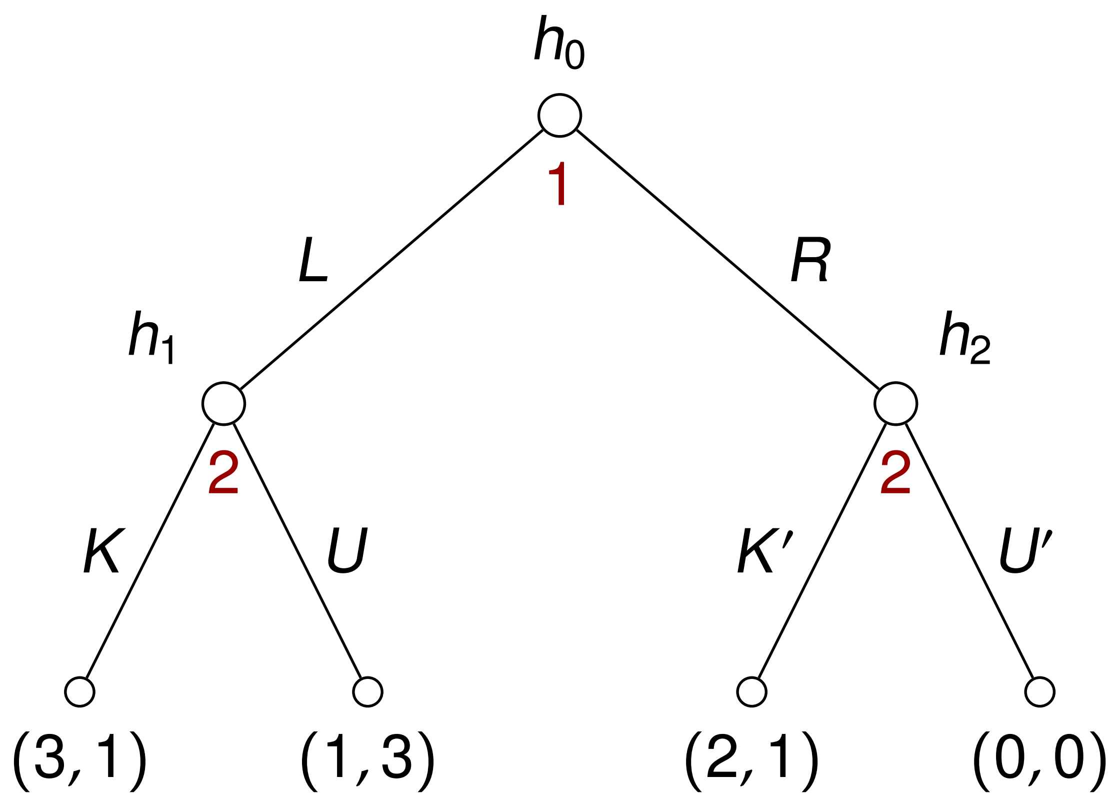
Is any strategy (strictly, weakly, very weakly) dominant?

Is any strategy (strictly, weakly, very weakly) dominated?

Is any strategy never best response?

Are there Nash equilibria in pure strategies ?

# Example




| | $KK'$ | $KU'$ | $UK'$ | $UU'$ |
|---|---|---|---|---|
| $L$ | $3, 1$ | $3, 1$ | $1, 3$ | $1, 3$ |
| $R$ | $2, 1$ | $0, 0$ | $2, 1$ | $0, 0$ |

Find all pure strategies of both players.

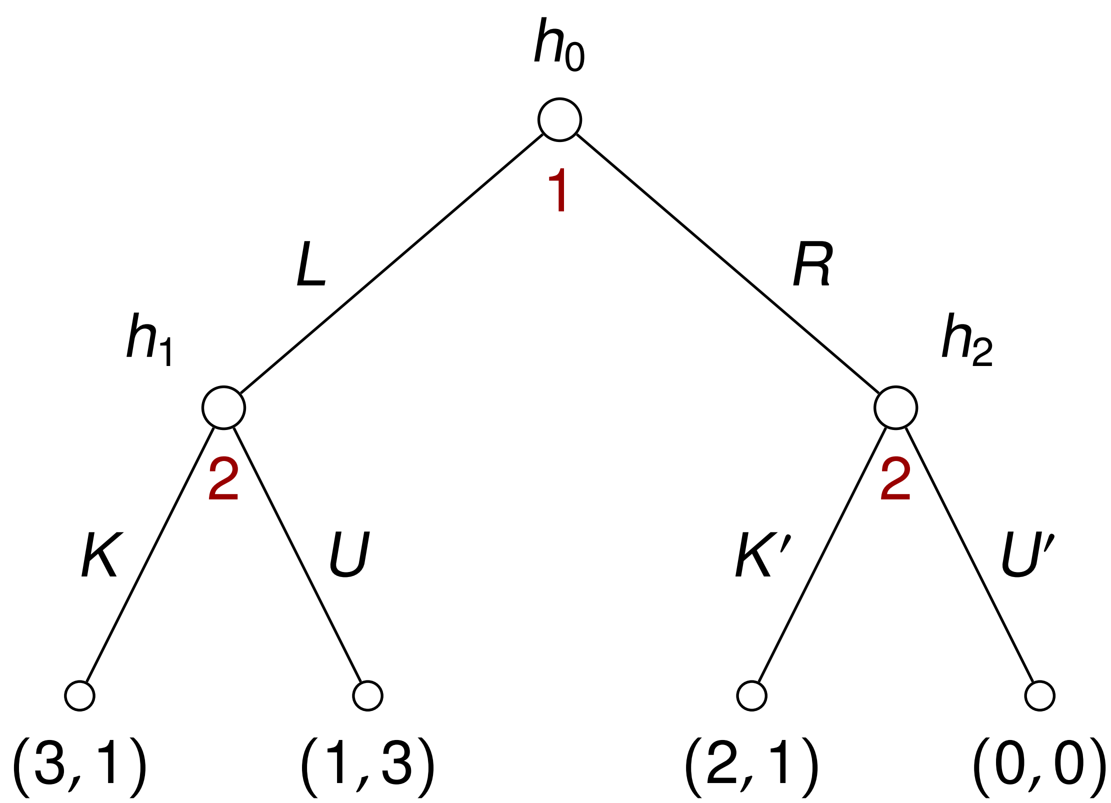
Is any strategy (strictly, weakly, very weakly) dominant?

Is any strategy (strictly, weakly, very weakly) dominated?

Is any strategy never best response?

Are there Nash equilibria in pure strategies ?

# Criticism of Nash Equilibria




| | $KK'$ | $KU'$ | $UK'$ | $UU'$ |
|---|---|---|---|---|
| $L$ | 3, 1 | 3, 1 | 1, 3 | 1, 3 |
| $R$ | 2, 1 | 0, 0 | 2, 1 | 0, 0 |

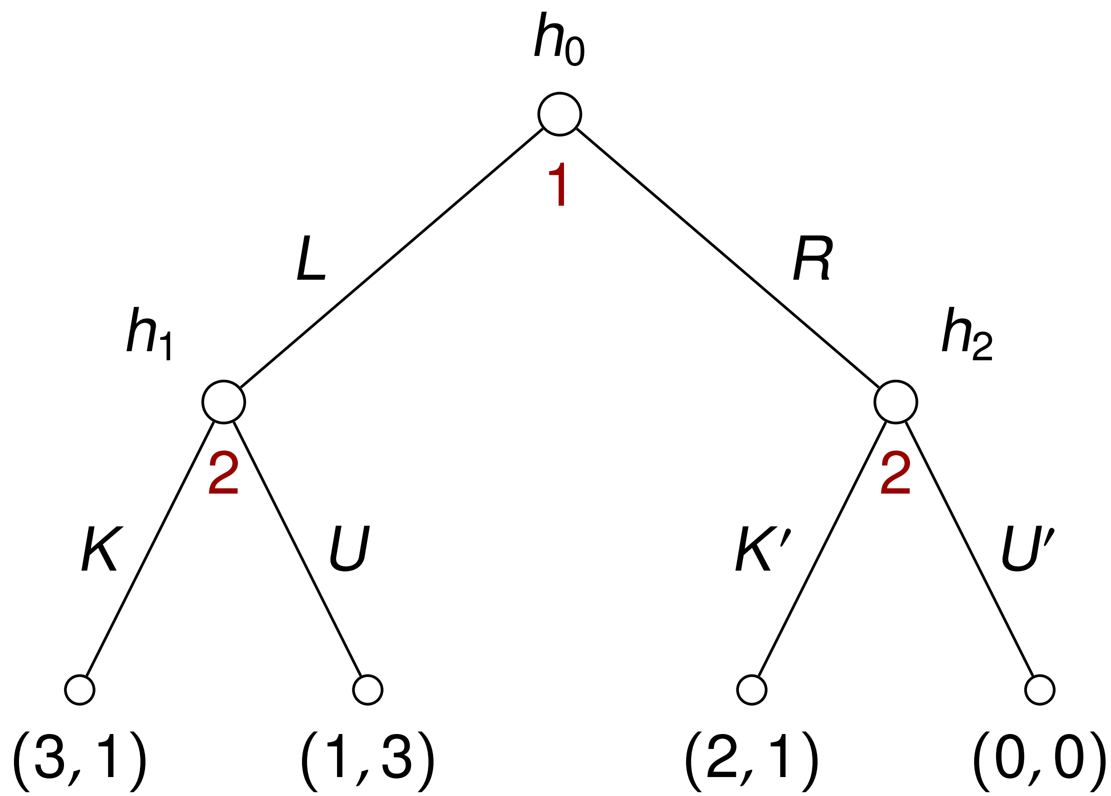
Two Nash equilibria in pure strategies: $(L, UU')$ and $(R, UK')$

Examine $(L, UU')$:

- Player 2 **threats** to play $U'$ in $h_2$,
- as a result, player 1 plays $L$,
- player 2 reacts to $L$ by playing the best response, i.e., $U$.

However, the threat is not *credible*, once a play reaches $h_2$, a rational player 2 chooses $K'$.

# Criticism of Nash Equilibria




| | $KK'$ | $KU'$ | $UK'$ | $UU'$ |
|---|---|---|---|---|
| $L$ | 3, 1 | 3, 1 | 1, 3 | 1, 3 |
| $R$ | 2, 1 | 0, 0 | 2, 1 | 0, 0 |

Two Nash equilibria in pure strategies: $(L, UU')$ and $(R, UK')$

Examine $(R, UK')$: This equilibrium is sensible in the following sense:

- Player 2 plays the best response in both $h_1$ and $h_2$
- Player 1 plays the "best response" in $h_0$ *assuming* that player 2 will play his best responses in the future.

This equilibrium is called *subgame perfect*.

# Subgame Perfect Equilibria

Given $h \in \mathcal{H}$, we denote by $\mathcal{H}^h$ the set of all nodes reachable from $h$.

**Definition 55 (Subgame)**

A *subgame* $G^h$ of $G$ rooted in $h \in \mathcal{H}$ is the restriction of $G$ to nodes reachable from $h$ in the game tree. More precisely, $G^h = (N, A, H^h, Z^h, \chi^h, \rho^h, \pi^h, h, u^h)$ where $H^h = H \cap \mathcal{H}^h$, $Z^h = Z \cap \mathcal{H}^h$, $\chi^h$ and $\rho^h$ are restrictions of $\chi$ and $\rho$ to $H^h$, resp.,

(Given a function $f : A \to B$ and $C \subseteq A$, a restriction of $f$ to $C$ is a function $g : C \to B$ such that $g(x) = f(x)$ for all $x \in C$.)

- $\pi^h$ is defined for $h' \in H^h$ and $a \in \chi^h(h')$ by $\pi^h(h', a) = \pi(h', a)$
- each $u_i^h$ is a restriction of $u_i$ to $Z^h$

**Definition 56**

A *subgame perfect equilibrium (SPE)* in pure strategies is a pure strategy profile $s \in S$ such that for any subgame $G^h$ of $G$, the restriction of $s$ to $H^h$ is a Nash equilibrium in pure strategies in $G^h$.

A restriction of $s = (s_1, \ldots, s_n) \in S$ to $H^h$ is a strategy profile $s^h = (s_1^h, \ldots, s_n^h)$ where $s_i^h(h') = s_i(h')$ for all $i \in N$ and all $h' \in H_i \cap H^h$.

# Stackelberg Competition – SPE

- $N = \{1, 2\}$, $A = [0, \infty)$
- $H = \{h_0, h_1^{q_1} \mid q_1 \in [0, \infty)\}$, $Z = \{z^{q_1, q_2} \mid q_1, q_2 \in [0, \infty)\}$
- $\chi(h_0) = [0, \infty)$, $\chi(h_1^{q_1}) = [0, \infty)$, $\rho(h_0) = 1$, $\rho(h_1^{q_1}) = 2$
- $\pi(h_0, q_1) = h_1^{q_1}$, $\pi(h_1^{q_1}, q_2) = z^{q_1, q_2}$
- The payoffs are $u_1(z^{q_1, q_2}) = q_1(\kappa - c - q_1 - q_2)$, $u_2(z^{q_1, q_2}) = q_2(\kappa - c - q_1 - q_2)$

Denote $\theta = \kappa - c$

---

Player 1 chooses $q_1$, we know that the best response of player 2 is $q_2 = (\theta - q_1)/2$ where $\theta = \kappa - c$.

Then $u_1(z^{q_1, q_2}) = q_1(\theta - q_1 - \theta/2 - q_1/2) = (\theta/2)q_1 - q_1^2/2$ which is maximized by $q_1 = \theta/2$, giving $q_2 = \theta/4$.

Then $u_1(z^{q_1, q_2}) = \theta^2/8$ and $u_2(z^{q_1, q_2}) = \theta^2/16$.

Note that firm 1 has an advantage as a leader.

# Existence of SPE

From this moment on we consider only **finite games**!

**Theorem 57**

*Every finite perfect-information extensive-form game has a SPE in pure strategies.*

**Proof:** By induction on the number of nodes.
**Base case:** If $|\mathcal{H}| = 1$, the only node is terminal, and the trivial pure strategy profile is SPE.
**Induction step:** Consider a game with more than one node. Let $K = \{h_1, \ldots, h_k\}$ be the set of all children of the root $h_0$.

By induction, for every $h_\ell$ there is a SPE $s^{h_\ell}$ in $G^{h_\ell}$.

For every $i \in N$, define a strategy $s_i$ of player $i$ in $G$ as follows:

- for $i = \rho(h_0)$ we have $s_i(h_0) \in \text{argmax}_{h_\ell \in K} u_i^{h_\ell}(s^{h_\ell})$
- for all $i \in N$ and $h \in H$ we have $s_i(h) = s_i^{h_\ell}(h)$ where $h \in H^{h_\ell} \cap H_i$

We claim that $s = (s_1, \ldots, s_n)$ is a SPE in pure strategies.

By definition, $s$ is NE in all subgames except (possibly) the $G$ itself.

□

# Existence of SPE (Cont.)

Let $h_\ell = s_{\rho(h_0)}(h_0)$.

Consider a possible deviation of player $i$.

Let $\bar{s}$ be another pure strategy profile in $G$ obtained from $s = (s_1, \ldots, s_n)$ by changing $s_i$.

First, assume that $i \neq \rho(h_0)$. Then

$$u_i(s) = u_i^{h_\ell}(s^{h_\ell}) \geq u_i^{h_\ell}(\bar{s}^{h_\ell}) = u_i(\bar{s})$$

Here the first equality follows from $h_\ell = s_{\rho(h_0)}(h_0)$ and that $s$ behaves similarly as $s^{h_\ell}$ in $G^{h_\ell}$, the inequality follows from the fact that $s^{h_\ell}$ is a NE in $G^{h_\ell}$, and the second equality follows from $h_\ell = s_{\rho(h_0)}(h_0) = \bar{s}_{\rho(h_0)}(h_0)$.

Second, assume that $i = \rho(h_0)$.

Let $h_r = \bar{s}_i(h_0) = \bar{s}_{\rho(h_0)}(h_0)$.

Then $u_i^{h_\ell}(s^{h_\ell}) \geq u_i^{h_r}(s^{h_r})$ because $h_\ell$ maximizes the payoff of player $i = \rho(h_0)$ in the children of $h_0$.

But then

$$u_i(s) = u_i^{h_\ell}(s^{h_\ell}) \geq u_i^{h_r}(s^{h_r}) \geq u_i^{h_r}(\bar{s}^{h_r}) = u_i(\bar{s})$$

# Backward Induction

The proof of Theorem 57 gives an efficient procedure for computing SPE for finite perfect-information extensive-form games.

**Backward Induction:** We inductively "attach" to every node $h$ a SPE $s^h$ in $G^h$, together with a vector of expected payoffs $u(h) = (u_1(h), \dots, u_n(h))$.

- **Initially:** Attach to each terminal node $z \in Z$ the empty profile $s^z = (\emptyset, \dots, \emptyset)$ and the payoff vector $u(z) = (u_1(z), \dots, u_n(z))$.
- **While**(there is an unattached node $h$ with all children attached):
  1. Let $K$ be the set of all children of $h$
  2. Let
  $$h_{max} \in \operatorname*{argmax}_{h' \in K} u_{\rho(h)}(h')$$
  3. Attach to $h$ a SPE $s^h$ where
     - $s^h_{\rho(h)}(h) = h_{\max}$
     - for all $i \in N$ and all $h' \in H_i$ define $s^h_i(h') = s^{\bar{h}}_i(h')$ where
     $h' \in H^{\bar{h}} \cap H_i$ (in $G^{\bar{h}}$, each $s^h_i$ behaves as $s^{\bar{h}}_i$ i.e. $\left(s^h\right)^{\bar{h}} = s^{\bar{h}}$)
  4. Attach to $h$ the expected payoffs $u_i(h) = u_i(h_{\max})$ for $i \in N$.

# Chess

Recall that in the model of chess, the payoffs were from $\{1, 0, -1\}$ and $u_1 = -u_2$ (i.e. it is zero-sum).

By Theorem 57, there is a SPE in pure strategies $(s_1^*, s_2^*)$.

However, then one of the following holds:

1. White has a winning strategy
   If $u_1(s_1^*, s_2^*) = 1$ and thus $u_2(s_1^*, s_2^*) = -1$
2. Black has a winning strategy
   If $u_1(s_1^*, s_2^*) = -1$ and thus $u_2(s_1^*, s_2^*) = 1$
3. Both players have strategies to force a draw
   If $u_1(s_1^*, s_2^*) = 0$ and thus $u_2(s_1^*, s_2^*) = 0$

**Question:** Which one is the right answer?
**Answer:** Nobody knows yet ... the tree is too big!
Even with $\sim$ 200 depth & $\sim$ 5 moves per node: $5^{200}$ nodes!