# Efficient Algorithms for Pure Nash Equilibria

In the step 2. of the backward induction, the algorithm may choose *an arbitrary* $h_{\max} \in \mathrm{argmax}_{h' \in K}\, u_{\rho(h)}(h')$ and always obtain a SPE.

In order to compute all SPE, the algorithm may systematically search through all possible choices of $h_{\max}$ throughout the induction.

Backward induction is too inefficient (unnecessarily searches through the whole tree).

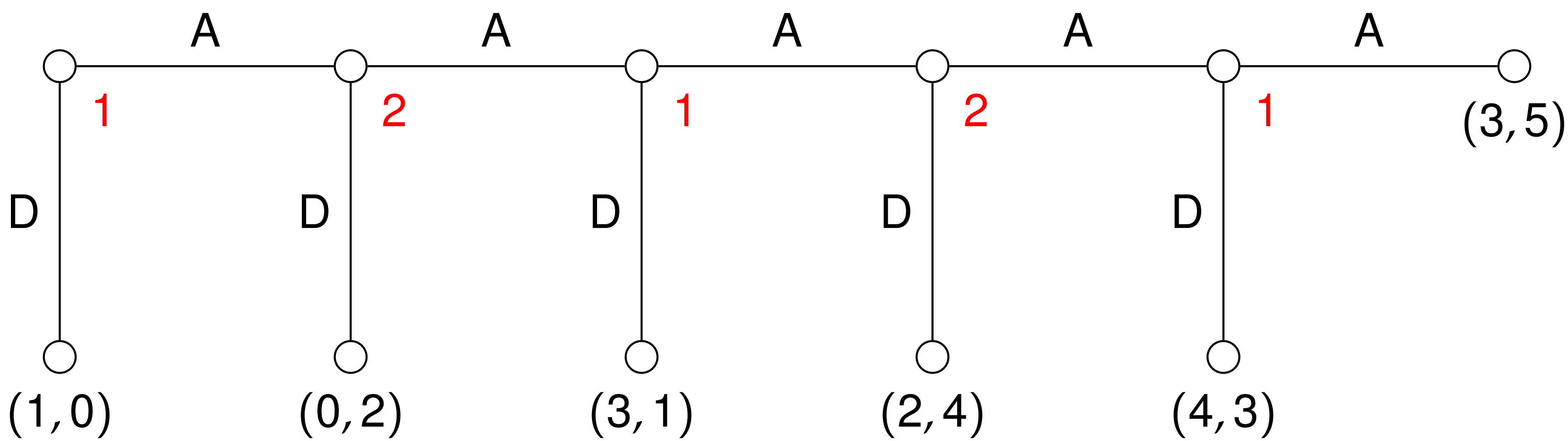
There are better algorithms, such as $\alpha$–$\beta$-prunning.

For details, extensions etc. see e.g.

- PB016 Artificial Intelligence I
- Multi-player alpha-beta prunning, R. Korf, *Artificial Intelligence 48*, pages 99-111, 1991
- Artificial Intelligence: A Modern Approach (3rd edition), S. Russell and P. Norvig, *Prentice Hall*, 2009

# Example

Centipede game:




SPE in pure strategies: $(DDD, DD)$ ... Isn't it weird?

There are serious issues here ...

- In laboratory setting, people usually play $A$ for several steps.
- There is a theoretical problem: Imagine, that you are player 2. What would you do when player 1 chooses $A$ in the first step? The SPE analysis says that you should go down, but the same analysis also says that the situation you are in cannot appear :-)

# Dynamic Games of Complete Information

## Extensive-Form Games

## **Mixed and Behavioral Strategies**

# Mixed and Behavioral Strategies

**Definition 58**

A *mixed strategy* $\sigma_i$ of player $i$ in $G$ is a mixed strategy of player $i$ in the corresponding strategic-form game.

I.e., a mixed strategy $\sigma_i$ of player $i$ in $G$ is a probability distribution on $S_i$ (recall that $S_i$ is the set of all pure strategies, i.e., functions of the form $s_i : H_i \to A$).

As before, we denote by $\sigma_i$ the set of all mixed strategies of player $i$ and by $\Sigma$ the set of all mixed strategy profiles $\Sigma_1 \times \cdots \times \Sigma_n$.

**Definition 59**

A *behavioral strategy* of player $i$ in $G$ is a function $\beta_i : H_i \to \Delta(A)$ such that for every $h \in H_i$ we have that $supp(\beta_i(h)) \subseteq \chi(h)$.

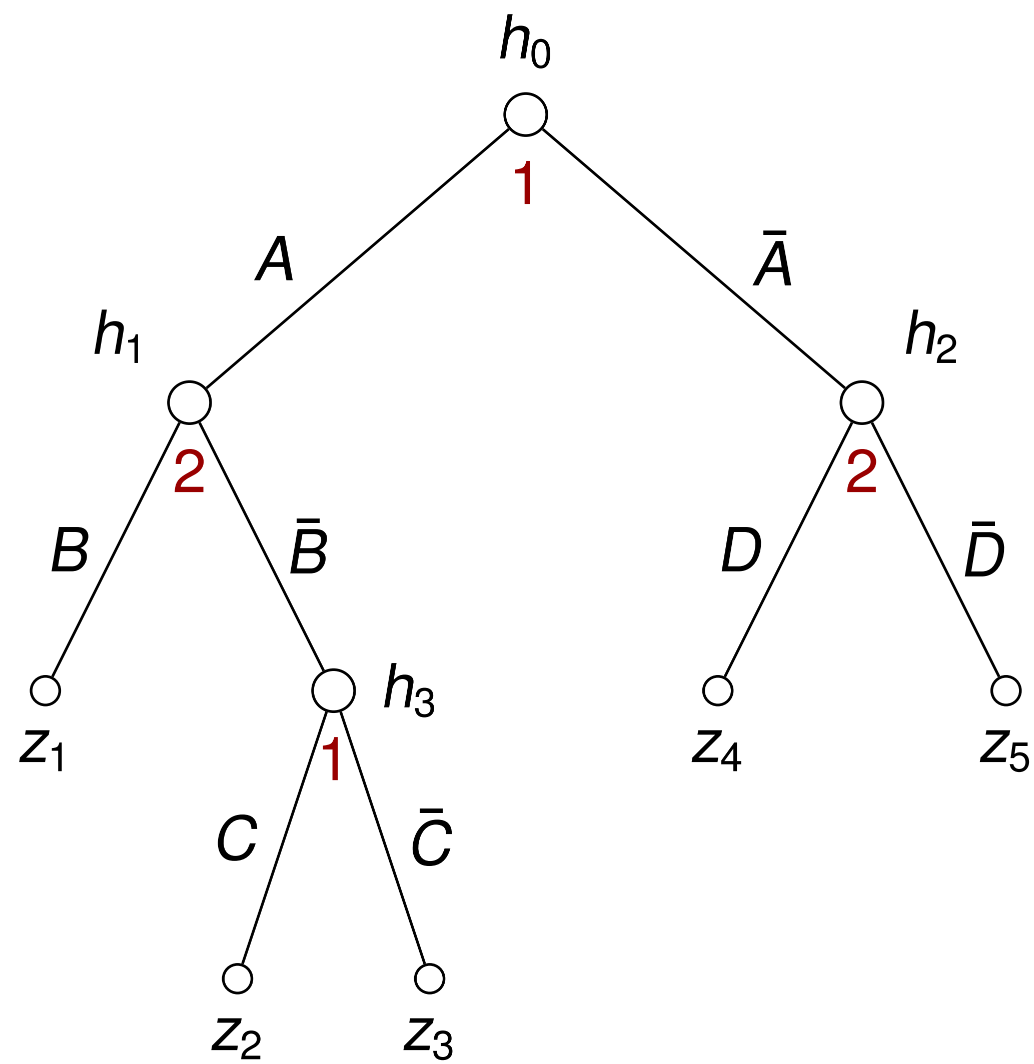
Given a profile $\beta = (\beta_1, \ldots, \beta_n)$ of behavioral strategies, we denote by $P_\beta(z)$ the probability of reaching $z \in Z$ when $\beta$ is used, i.e.,

$$P_\beta(z) = \prod_{\ell=1}^{k} \beta_{\rho(h_{\ell-1})}(h_\ell)(a_\ell)$$

where $h_0 a_1 h_1 a_2 h_2 \cdots a_k h_k$ is the unique path from $h_0$ to $h_k = z$.

We define $u_i(\beta) := \sum_{z \in Z} P_\beta(z) \cdot u_i(z)$.

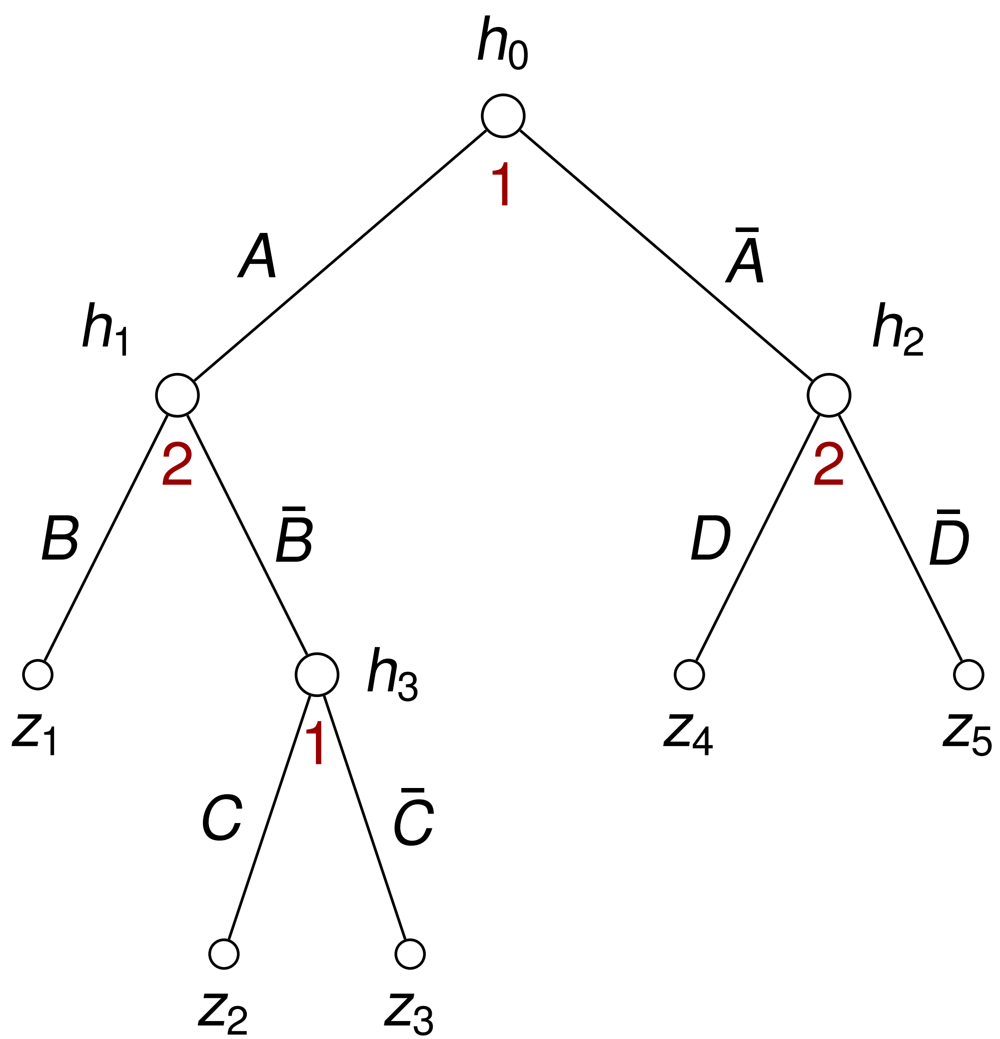
# Behavioral Strategies: Example




Pure strategies of player 1: $AC$, $A\bar{C}$, $\bar{A}C$, $\bar{A}\bar{C}$

An example of a mixed strategy $\sigma_1$ of player 1:
$\sigma_1(AC) = \frac{1}{3}$, $\sigma_1(A\bar{C}) = \frac{1}{9}$, $\sigma_1(\bar{A}C) = \frac{1}{6}$ and $\sigma_1(\bar{A}\bar{C}) = \frac{11}{18}$

# Behavioral Strategies: Example

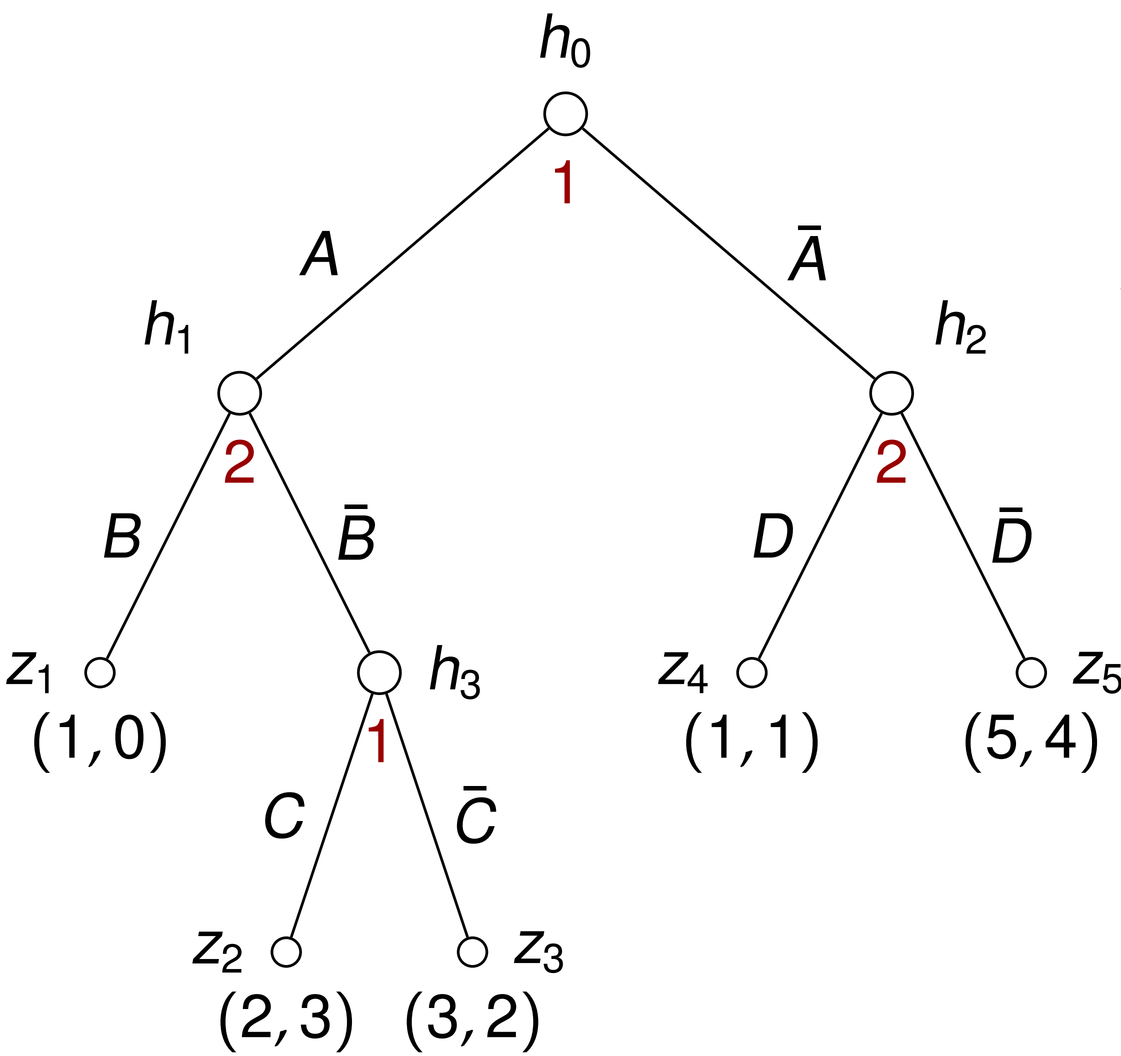



An example of behavioral strategies of both players:

- player 1: $\beta_1(h_0)(A) = \frac{1}{3}$ and $\beta_1(h_3)(C) = \frac{1}{2}$
- player 2: $\beta_2(h_1)(B) = \frac{1}{4}$ and $\beta_2(h_2)(D) = \frac{1}{5}$

$P_{(\beta_1,\beta_2)}(z_2) = \frac{1}{3}\left(1 - \frac{1}{4}\right)\frac{1}{2} = \frac{1}{8}$

# Behavioral Strategies: Example




$\beta = (\beta_1, \beta_2)$

- player 1: $\beta_1(h_0)(A) = \frac{1}{3}$ and $\beta_1(h_3)(C) = \frac{1}{2}$
- player 2: $\beta_2(h_1)(B) = \frac{1}{4}$ and $\beta_2(h_2)(D) = \frac{1}{5}$

$$
\begin{aligned}
u_1(\beta) &= P_\beta(z_1) \cdot 1 + P_\beta(z_2) \cdot 2 + P_\beta(z_3) \cdot 3 + P_\beta(z_4) \cdot 1 + P_\beta(z_5) \cdot 5 \\
&= \frac{1}{3}\frac{1}{4}1 + \frac{1}{3}\frac{3}{4}\frac{1}{2}2 + \frac{1}{3}\frac{3}{4}\frac{1}{2}3 + \frac{2}{3}\frac{1}{5}1 + \frac{2}{3}\frac{4}{5}5 \approx 3.508
\end{aligned}
$$

# Mixed/Behavioral Profiles

Each pure strategy can be considered as a behavioral strategy.

**Definition 60**

A *mixed/behavioral strategy profile* is a tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each $\alpha_i$ is either a mixed, or a behavioral strategy.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a mixed/behavioral strategy profile, and let $M = \{i_1, \ldots, i_k\} \subseteq N$ be the set of all players $i_j \in N$ such that *$\alpha_{i_j}$ is a mixed strategy*. We define

$$u_i(\alpha) = \sum_{s_{i_1} \in S_{i_1}} \cdots \sum_{s_{i_k} \in S_{i_k}} \left( \prod_{\ell=1}^{k} \alpha_{i_\ell}(s_{i_\ell}) \right) \cdot u_i(\alpha'_1, \ldots, \alpha'_n)$$

where $\alpha'_j = \begin{cases} s_j & \text{if } j \in M, \\ \alpha_j & \text{otherwise.} \end{cases}$

Intuitively, $u_i(\alpha)$ is the expected payoff of player $i$ in the following play: First, each player $i_\ell \in M$ chooses his pure strategy $s_{i_\ell}$ randomly with the probability $\alpha_{i_\ell}(s_{i_\ell})$, then these fixed pure strategies are played against the behavioral strategies of players from $N \setminus M$ (who may still randomize along the play).

# Equivalence of Mixed and Behavioral Strategies

We show how to translate behavioral strategies to equivalent mixed ones (w.r.t. probabilities of reaching terminal nodes) and vice versa.

**Behavioral to mixed:** We say that a mixed strategy $\sigma_i$ is *induced by* a behavioral strategy $\beta_i$ if

$$\sigma_i(s_i) = \prod_{h \in H_i} \beta_i(h)(s_i(h)) \text{ for all } s_i \in S_i$$

**Mixed to behavioral:** For this direction some notation is needed. Given $h \in \mathcal{H}$, we denote by $w[h]$ the unique path from $h_0$ to $h$.

Given $h \in H_i$, we denote by $S_i^h$ the set of all pure strategies $s_i \in S_i$ such that for every $h' \in H_i$ visited by $w[h]$ we have that $s_i(h')$ is the action chosen in $h'$ on $w[h]$.

Intuitively, $S_i^h$ consists of all pure strategies that on the unique path from $h_0$ to $h$ chose the appropriate actions to stay on the path. In other words, $h$ can be reached using $s_i$ (assuming that the opponents play appropriately) iff $s_i \in S_i^h$.

Given $h \in H_i$ and $a \in \chi(h)$, we denote by $S_i^{h,a} \subseteq S_i^h$ the set of all pure strategies $s_i \in S_i^h$ such that $s_i(h) = a$.

I.e., strategies of $S_i^{h,a}$ may reach $h$ and then choose $a$ there.

# Equivalence of Mixed and Behavioral Strategies (Cont.)

We say that a behavioral strategy $\beta_i$ is *induced by* a mixed strategy $\sigma_i$ if the following holds:

For every $h \in H_i$ and $a \in \chi(h)$

- either

$$\sum_{s_i \in S_i^h} \sigma_i(s_i) = 0$$

- or

$$\beta_i(h)(a) = \frac{\sum_{s_i \in S_i^{h,a}} \sigma_i(s_i)}{\sum_{s_i \in S_i^h} \sigma_i(s_i)}$$

Intuitively, $\beta_i(h)(a)$ is the probability of selecting $a$ in $h$ assuming that $h$ can be reached with a positive probability if the other players play appropriately.

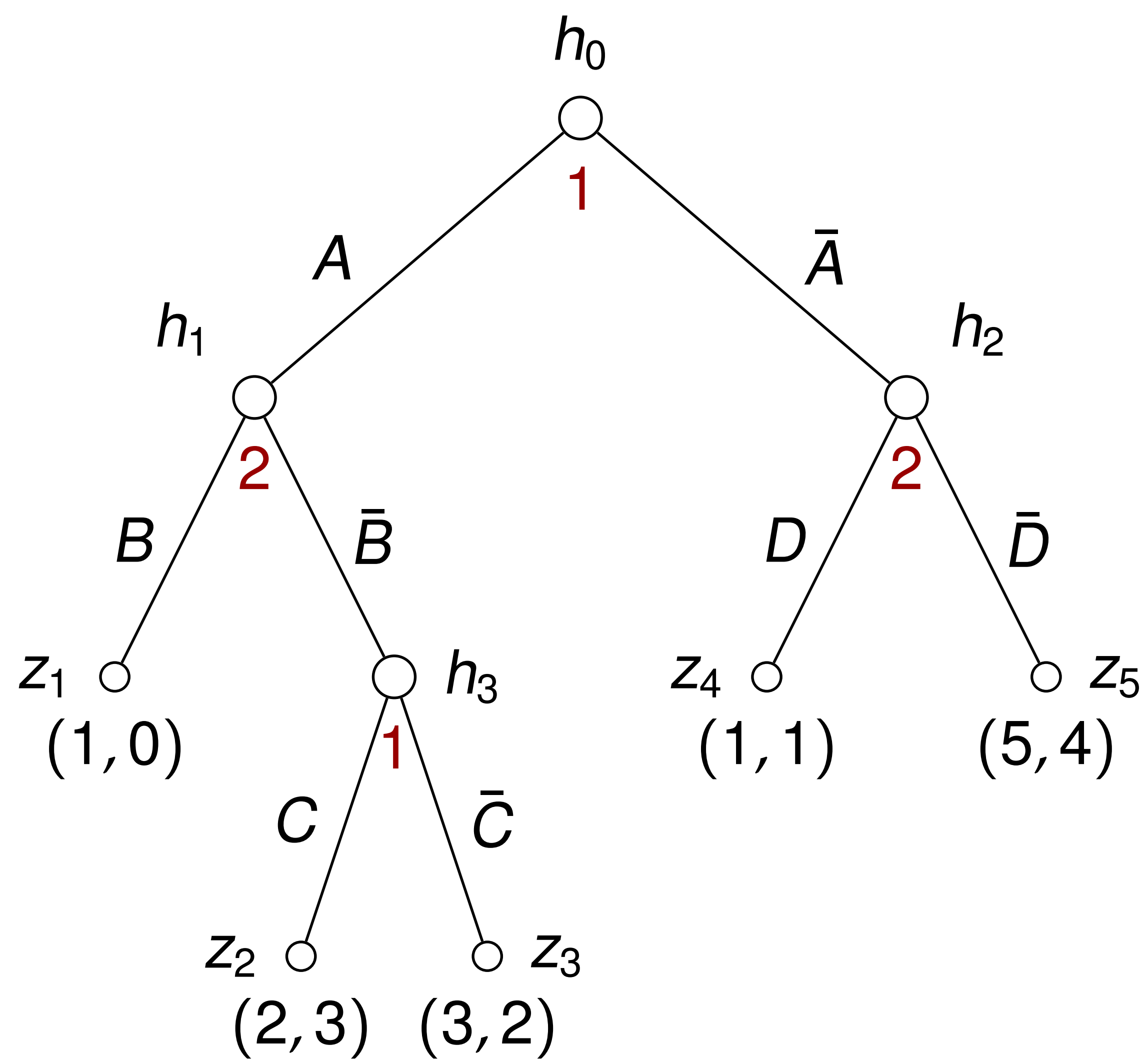
If the probability of reaching $h$ using $\sigma_i$ is zero (no matter of what the opponents are doing), then the $\beta_i(h)$ may be defined arbitrarily since $h$ is reached with zero probability using $\beta$ as well.

# Equivalence of Mixed and Behavioral Strategies

**Theorem 61**
*Let $\alpha$ be a mixed/behavioral strategy profile and let $\alpha'$ be any mixed/behavioral profile obtained from $\alpha$ by substituting some of the strategies in $\alpha$ with strategies they induce. Then $u_i(\alpha) = u_i(\alpha')$.*

In fact, any node of $\mathcal{H}$ is reached from $h_0$ with the same probability for all such $\alpha'$.

# Equivalence of Mixed and Behavioral Strategies




Pure strategies of player 1: $AC, A\bar{C}, \bar{A}C, \bar{A}\bar{C}$

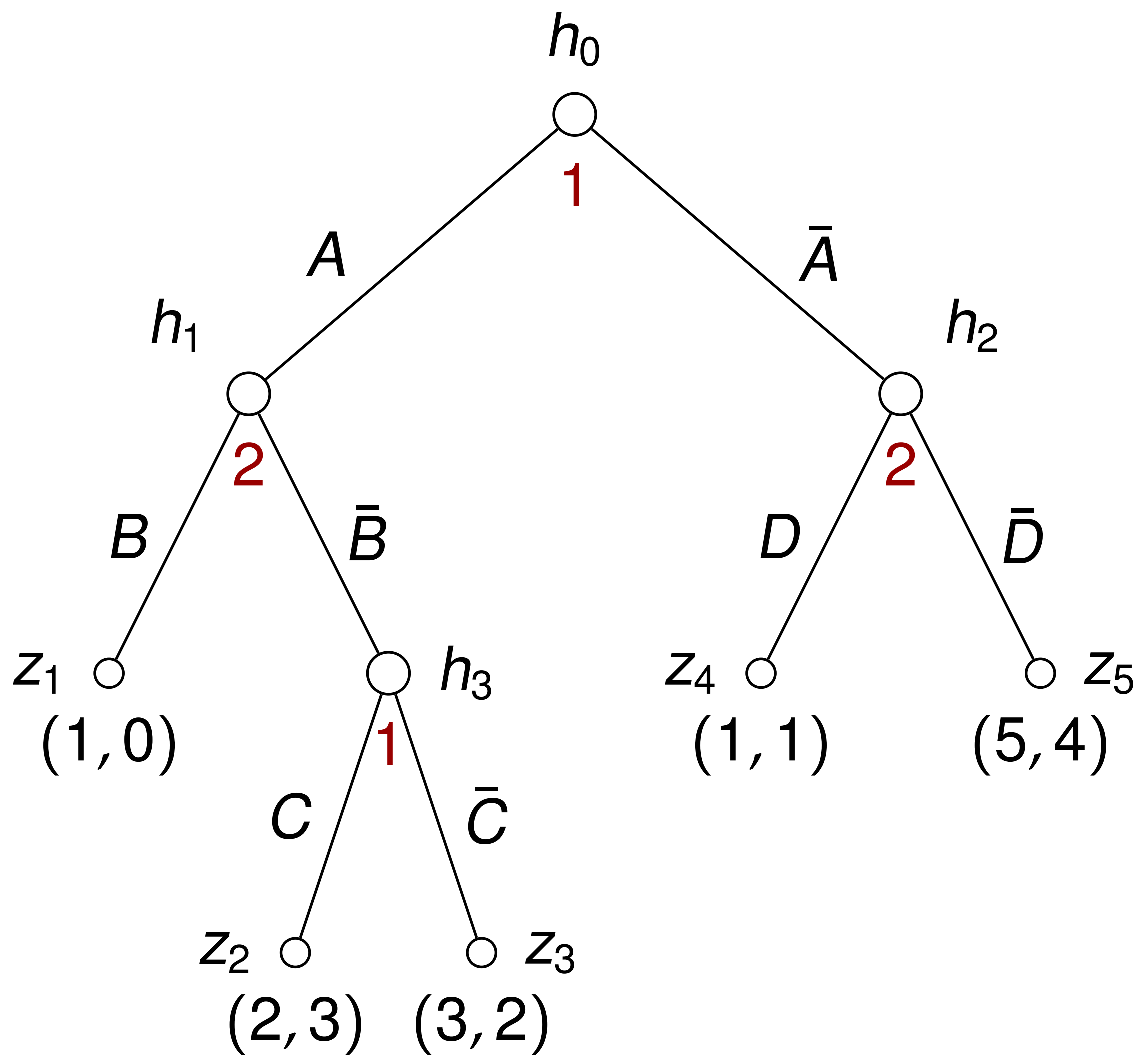
Pure strategies of player 2: $BD, B\bar{D}, \bar{B}D, \bar{B}\bar{D}$

Mixed strategies of player 1: $\sigma_1 = (p_{AC}, p_{A\bar{C}}, p_{\bar{A},C}, p_{\bar{A}\bar{C}})$

(Here $p_{XY} = \sigma_1(s)$ where $s$ is a pure str. such that $s(h_0) = X$, $s(h_3) = Y$)

Mixed strategies of player 2: $\sigma_2 = (p_{BD}, p_{B\bar{D}}, p_{\bar{B}D}, p_{\bar{B}\bar{D}})$

# Equivalence of Mixed and Behavioral Strategies




Behavioral strategies of player 1: $\beta_1 = (q_A, q_C)$ were $q_A = \beta_1(h_0)(A)$ and $q_C = \beta_1(h_3)(C)$; Denote $q_{\bar{A}} = 1 - q_A$ and $q_{\bar{C}} = 1 - q_C$

Behavioral strategies of player 2: $\beta_2 = (q_B, q_D)$ and we use $q_{\bar{B}} = 1 - q_B$ and $q_{\bar{D}} = 1 - q_D$

# Equivalence of Mixed and Behavioral Strategies

Behavioral to mixed: Given $\beta_1 = (q_A, q_C)$ and $\beta_2 = (q_B, q_D)$ define

$$\sigma_1 = (p_{AC}, p_{A\bar{C}}, p_{\bar{A},C}, p_{\bar{A}\bar{C}}) := (q_A q_C, q_A q_{\bar{C}}, q_{\bar{A}} q_C, q_{\bar{A}} q_{\bar{C}})$$

$$\sigma_2 = (p_{BD}, p_{B\bar{D}}, p_{\bar{B}D}, p_{\bar{B}\bar{D}}) := (q_B q_D, q_B q_{\bar{D}}, q_{\bar{B}} q_D, q_{\bar{B}} q_{\bar{D}})$$

What is the probability of reaching $z_2$ ?

- Using $(\beta_1, \beta_2)$ : $q_A q_{\bar{B}} q_C$
  (i.e. multiply the probabilities assigned by $\beta_1, \beta_2$ along the path from $h_0$ to $z_2$)
- Using $(\sigma_1, \sigma_2)$ : $(q_A q_C)(q_{\bar{B}} q_D + q_{\bar{B}} q_{\bar{D}}) = q_A q_{\bar{B}} q_C$
  (i.e., player 1 needs to choose the pure strategy $AC$, player 2 needs to choose any pure strategy which selects $\bar{B}$)
- Using $(\sigma_1, \beta_2)$ : $(q_A q_C) q_{\bar{B}} = q_A q_{\bar{B}} q_C$
  (i.e., first player 1 chooses a pure strategy, this needs to be $AC$, and then player 2 plays against this particular strategy by choosing $\bar{B}$)
- Using $(\beta_1, \sigma_2)$ : $(q_{\bar{B}} q_D + q_{\bar{B}} q_{\bar{D}}) q_A q_C = q_A q_{\bar{B}} q_C$
  (i.e., first player 2 chooses a pure strategy, needs to be one playing $\bar{B}$ in $h_1$, and then player 1 plays against this strategy by choosing $A$ and $C$)

# Equivalence of Mixed and Behavioral Strategies

Mixed to behavioral: Given $\sigma_1 = (p_{AC}, p_{A\bar{C}}, p_{\bar{A},C}, p_{\bar{A}\bar{C}})$ and $\sigma_2 = (p_{BD}, p_{B\bar{D}}, p_{\bar{B}D}, p_{\bar{B}\bar{D}})$ we have

- $\beta_1 = (q_A, q_C)$ where

$$q_A = p_{AC} + p_{A\bar{C}} \qquad\qquad q_C = \begin{cases} \frac{p_{AC}}{p_{AC}+p_{A\bar{C}}} & \text{if } p_{AC} + p_{A\bar{C}} > 0 \\ x & \text{otherwise} \end{cases}$$

  Here $x$ is an arbitrary number between 0 and 1.
- $\beta_2 = (q_B, q_D)$ where

$$q_B = p_{BD} + p_{B\bar{D}} \qquad\qquad q_D = p_{BD} + p_{\bar{B}D}$$

# Equivalence of Mixed and Behavioral Strategies

First, consider $q_A = p_{AC} + p_{A\bar{C}} > 0$.

What is the probability of reaching $z_2$ ?

- Using $(\sigma_1, \sigma_2)$ : $p_{AC} \cdot (p_{\bar{B}D} + p_{\bar{B}\bar{D}})$

  i.e., player 1 chooses $AC$ and player 2 chooses a pure str. playing $\bar{B}$

- Using $(\beta_1, \beta_2)$ :

$$
\begin{aligned}
q_A \cdot q_{\bar{B}} \cdot q_C &= (p_{AC} + p_{A\bar{C}}) \cdot q_{\bar{B}} \cdot \frac{p_{AC}}{p_{AC} + p_{A\bar{C}}} \\
&= q_{\bar{B}} \cdot p_{AC} \\
&= p_{AC} \cdot (1 - q_B) \\
&= p_{AC} \cdot (1 - (p_{BD} + p_{B\bar{D}})) \\
&= p_{AC} \cdot (p_{\bar{B}D} + p_{\bar{B}\bar{D}})
\end{aligned}
$$

- Using $(\beta_1, \sigma_2)$ :

$$
(p_{\bar{B}D} + p_{\bar{B}\bar{D}}) \cdot q_A \cdot q_C = q_A \cdot q_{\bar{B}} \cdot q_C = p_{AC} \cdot (p_{\bar{B}D} + p_{\bar{B}\bar{D}})
$$

  i.e., first player 2 chooses a pure strategy playing $\bar{B}$ in $h_1$ and then player 1 plays the behavioral strategy $\beta_1$ against it

# Equivalence of Mixed and Behavioral Strategies

- Using $(\sigma_1, \beta_2)$ : $p_{AC} \cdot q_{\bar{B}} = p_{AC} \cdot (p_{\bar{B}D} + p_{\bar{B}\bar{D}})$

  i.e., first player 1 chooses the pure strategy *AC* and then player 2 plays the behavioral str. $\beta_2$ against it

Observe that all possible combinations of mixed and behavioral strategies give the same probability of reaching $z_2$; this holds for all terminal nodes and hence all combinations give the same payoff.

Now, assume $q_A = p_{AC} + p_{A\bar{C}} = 0$ (which implies $p_{AC} = 0$).

What is the probability of reaching $z_2$ ?

- Using $(\sigma_1, \sigma_2)$ : $p_{AC} \cdot (p_{\bar{B}D} + p_{\bar{B}\bar{D}}) = 0$
- Using $(\beta_1, \beta_2)$ : $q_A \cdot q_{\bar{B}} \cdot q_C = 0$
- Using $(\beta_1, \sigma_2)$ : $(p_{\bar{B}D} + p_{\bar{B}\bar{D}}) \cdot q_A \cdot q_C = 0$
- Using $(\sigma_1, \beta_2)$ : $p_{AC} \cdot q_{\bar{B}} = 0$

# Behavioral (Mixed) Strategy SPE

Let us denote by $\mathcal{B}_i$ the set of all behavioral strategies of player $i$, and by $\mathcal{B}$ the set of all behavioral strategy profiles $\mathcal{B}_1 \times \ldots \times \mathcal{B}_n$.

**Definition 62**

$\beta = (\beta_1, \ldots, \beta_n) \in \mathcal{B}$ is a behavioral Nash equilibrium if

$$u_i(\beta_i, \beta_{-i}) \geq u_i(\beta_i', \beta_{-i}) \qquad \text{for all } i \in N \text{ and } \beta_i' \in \mathcal{B}_i$$

Observe that due to Theorem 61 behavioral NE coincide with mixed NE.

**Definition 63**

A *subgame perfect equilibrium (SPE)* in *behavioral* strategies is a behavioral strategy profile $\beta \in \mathcal{B}$ such that for any subgame $G^h$ of $G$, the restriction of $\beta$ to $H^h$ is a behavioral Nash equilibrium.

Here $\beta = (\beta_1, \ldots, \beta_n)$ and the restriction of $\beta$ to $G^h$ is a behavioral strategy profile $\beta^h = (\beta_1^h, \ldots, \beta_n^h)$ where each $\beta_i^h$ is a restriction of $\beta_i$ to $H^h \cap H_i$.

**Theorem 64**

*There exists a* **pure** *strategy profile which is a SPE in* behavioral *strategies.*

The proof is similar to the proof of Theorem 57.

# Comments on Algorithms

Note that some SPE in behavioral strategies can be computed using the backward induction.

Indeed, the algorithm computes a pure strategy profile where each player always maximizes his value; such a pure strategy profile is SPE in both pure and behavioral strategies.

Even though there always exists a pure SPE, there may exist (a continuum of) SPE composed of "non-pure" behavioral strategies.

However, the necessary and sufficient condition for existence of such SPE is that at some point of the backward induction one of the players (say $i$) has two or more alternatives with the same equilibrium payoff.

The same payoff is only for the player $i$, the other players may have different payoffs depending on the choice of the player $i$.

Then any convex combination of such alternatives can be made by the player $i$, still leading to SPE (of course, for each combination the resulting SPE may be different).

For two players the backward induction can be extended to compute (a finite representation of) all SPE in behavioral strategies in polynomial time.

# Dynamic Games of Complete Information

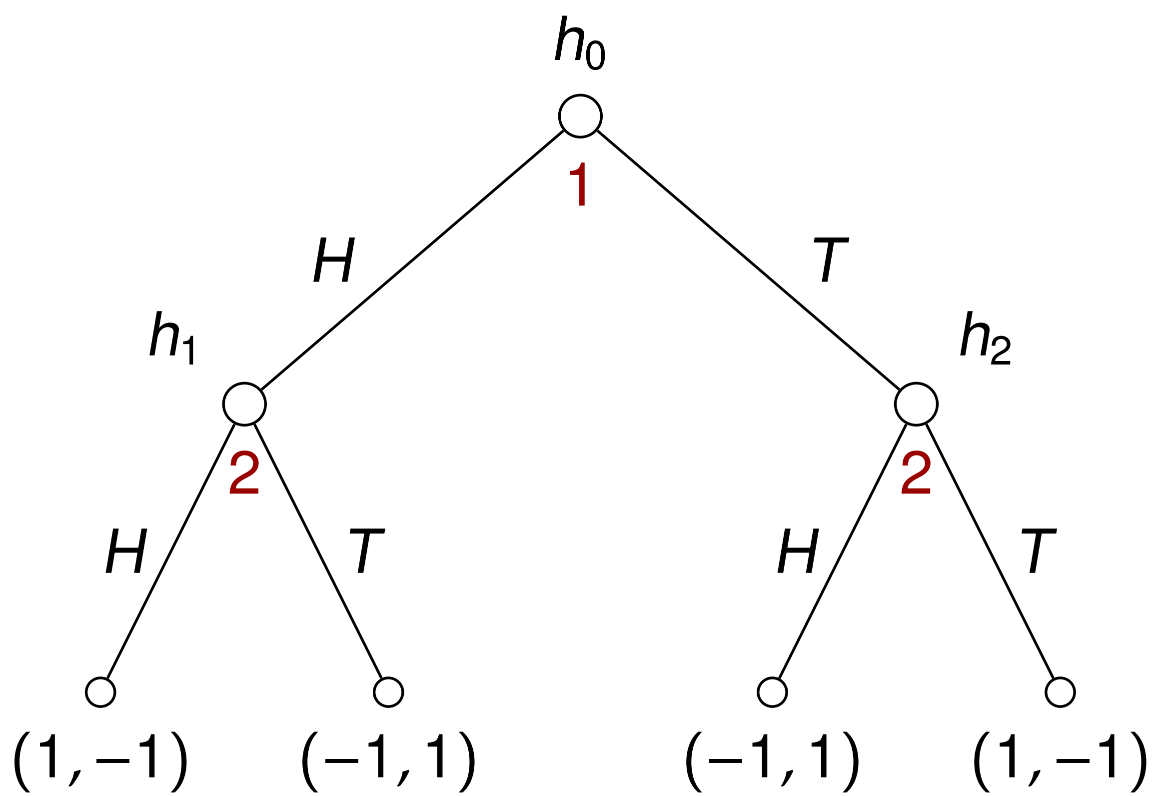
## Extensive-Form Games

## **Imperfect-Information Games**

# Extensive-form of Matching Pennies

Is it possible to model Matching pennies using extensive-form games?

| | $H$ | $T$ |
|---|---|---|
| $H$ | $1,-1$ | $-1,1$ |
| $T$ | $-1,1$ | $1,-1$ |



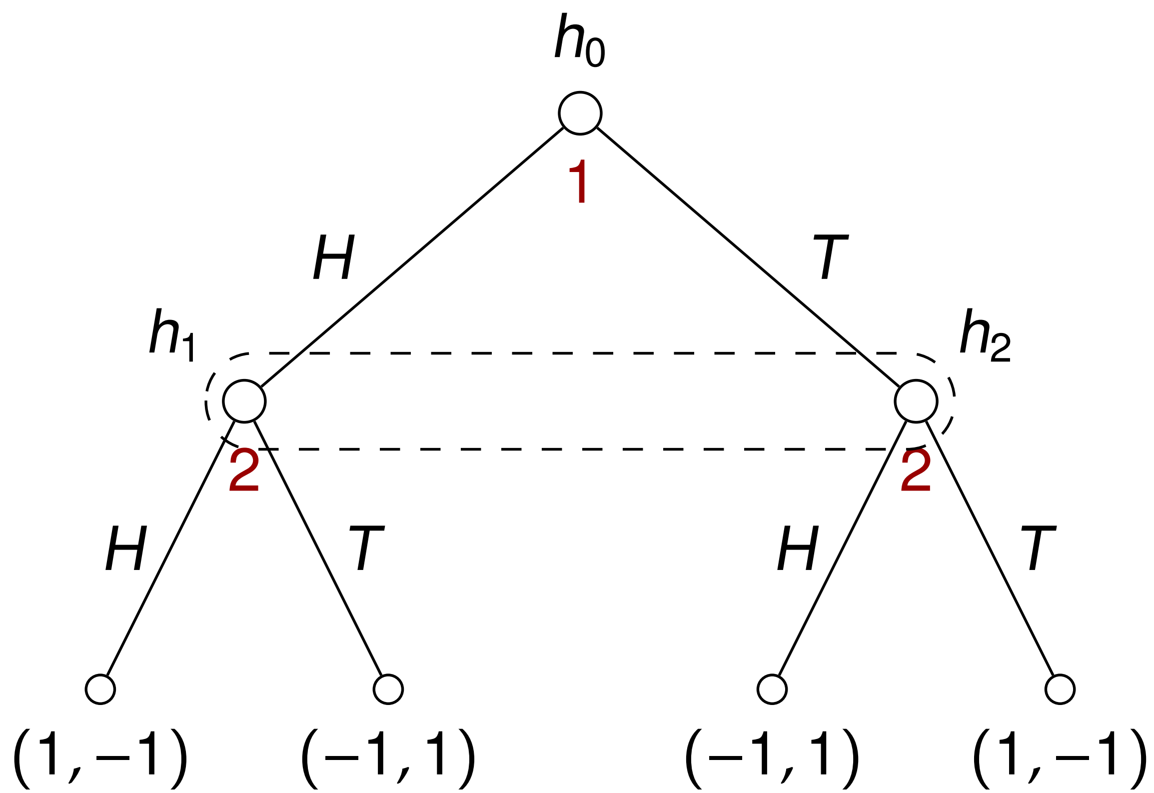

The problem is that player 2 is "perfectly" informed about the choice of player 1. In particular, there are pure Nash equilibria $(H, TH)$ and $(T, TH)$ in the extensive-form game as opposed to the strategic-form.

Reversing the order of players does not help.

We need to extend the formalism to be able to hide some information about previous moves.

# Extensive-form of Matching Pennies

Matching pennies can be modeled using
an *imperfect-information* extensive-form game:




Here $h_1$ and $h_2$ belong to the same *information set* of player 2.

As a result, player 2 is not able to distinguish between $h_1$ and $h_2$.

So even though players do not move simultaneously, the information player 2 has about the current situation is the same as in the simultaneous case.

# Imperfect Information Games

An *imperfect-information extensive-form game* is a tuple $G_{imp} = (G_{perf}, I)$ where

- $G_{perf} = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$ is a perfect-information extensive-form game (called *the underlying game*),
- $I = (I_1, \ldots, I_n)$ where for each $i \in N = \{1, \ldots, n\}$

  $$I_i = \{I_{i,1}, \ldots, I_{i,k_i}\}$$

  is a collection of *information sets* for player $i$ that satisfies
  - $\bigcup_{j=1}^{k_i} I_{i,j} = H_i$ and $I_{i,j} \cap I_{i,k} = \emptyset$ for $j \neq k$
    (i.e., $I_i$ is a partition of $H_i$)
  - for all $h, h' \in I_{i,j}$, we have $\rho(h) = \rho(h')$ and $\chi(h) = \chi(h')$
    (i.e., nodes from the same information set are owned by the same player and have the same sets of enabled actions)

Given an information set $I_{i,j}$, we denote by $\chi(I_{i,j})$ the set of all actions enabled in some (and hence all) nodes of $I_{i,j}$.

Now we define the set of pure, mixed, and behavioral strategies in $G_{imp}$ as subsets of pure, mixed, and behavioral strategies, resp., in $G_{perf}$ that respect the information sets.

# Imperfect Information Games – Strategies

Let $G_{imp} = (G_{perf}, I)$ be an imperfect-information extensive-form game where $G_{perf} = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$.

**Definition 65**

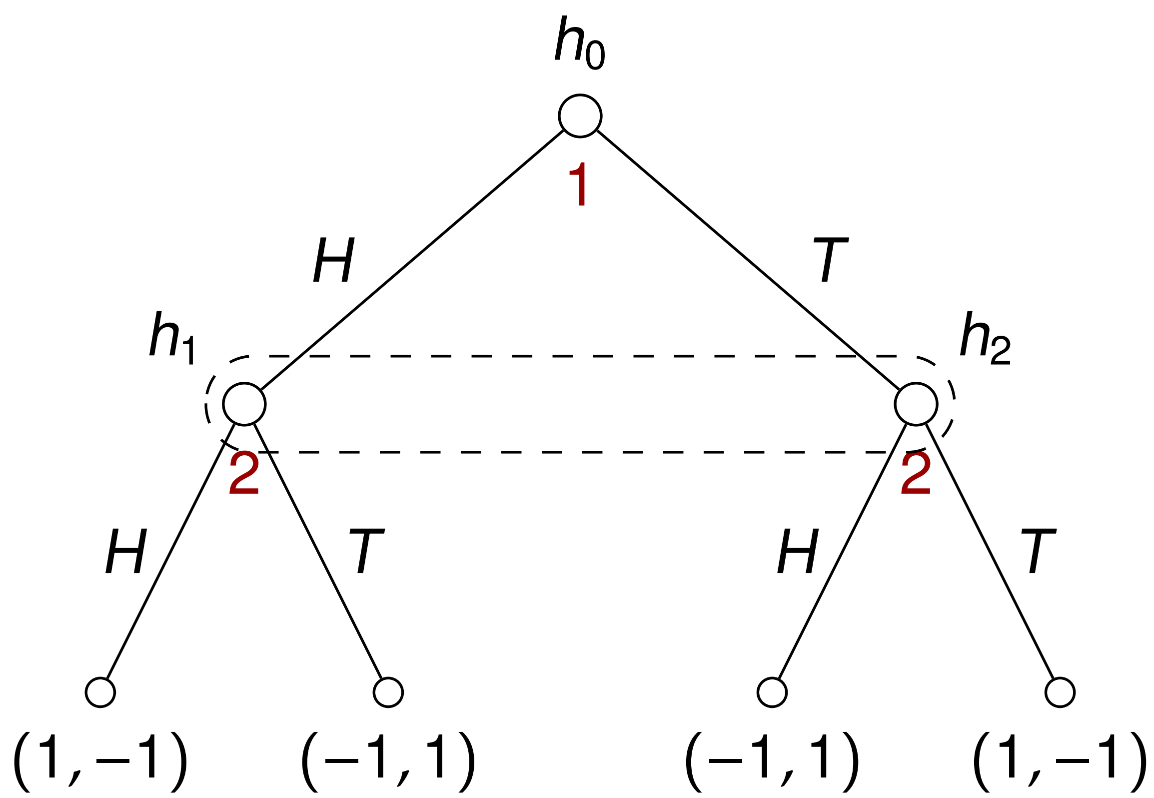
A *pure strategy* of player $i$ in $G_{imp}$ is a pure strategy $s_i$ in $G_{perf}$ such that for all $j = 1, \ldots, k_i$ and all $h, h' \in I_{i,j}$ holds $s_i(h) = s_i(h')$.

Note that each $s_i$ can also be seen as a function $s_i : I_i \to A$ such that for every $I_{i,j} \in I_i$ we have that $s_i(I_{i,j}) \in \chi(I_{i,j})$.

As before, we denote by $S_i$ the set of all pure strategies of player $i$ in $G_{imp}$, and by $S = S_1 \times \cdots \times S_n$ the set of all pure strategy profiles.

As in the perfect-information case we have a corresponding strategic-form game $\bar{G}_{imp} = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$.

# Matching Pennies




$I_1 = \{I_{1,1}\}$ where $I_{1,1} = \{h_0\}$

$I_1 = \{I_{2,1}\}$ where $I_{2,1} = \{h_1, h_2\}$

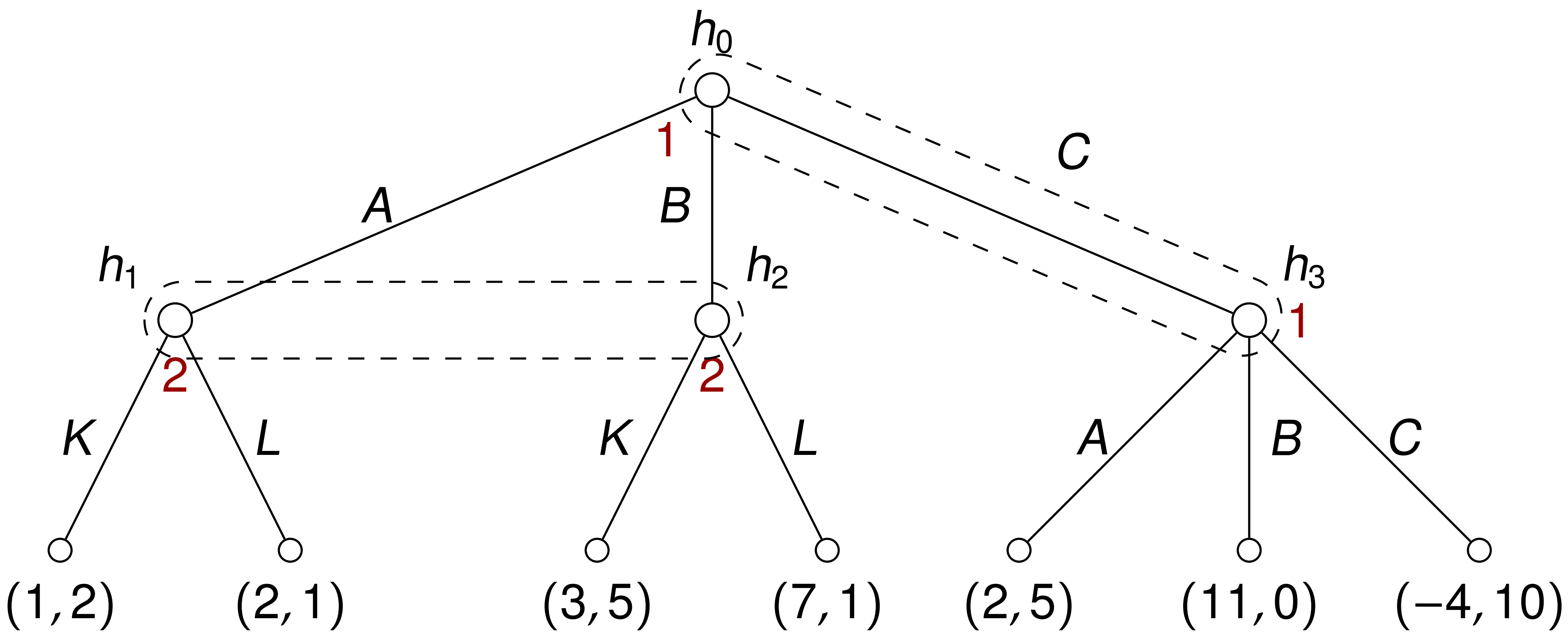
Example of pure strategies:

- $s_1(I_{1,1}) = H$ which describes the strategy $s_1(h_0) = H$
- $s_2(I_{2,1}) = T$ which describes the strategy $s_2(h_1) = s_2(h_2) = T$

  (it is also sufficient to specify $s_2(h_1) = T$ since then $s_2(h_2) = T$)

So we really have strategies $H, T$ for player 1 and $H, T$ for player 2.

# Weird Example

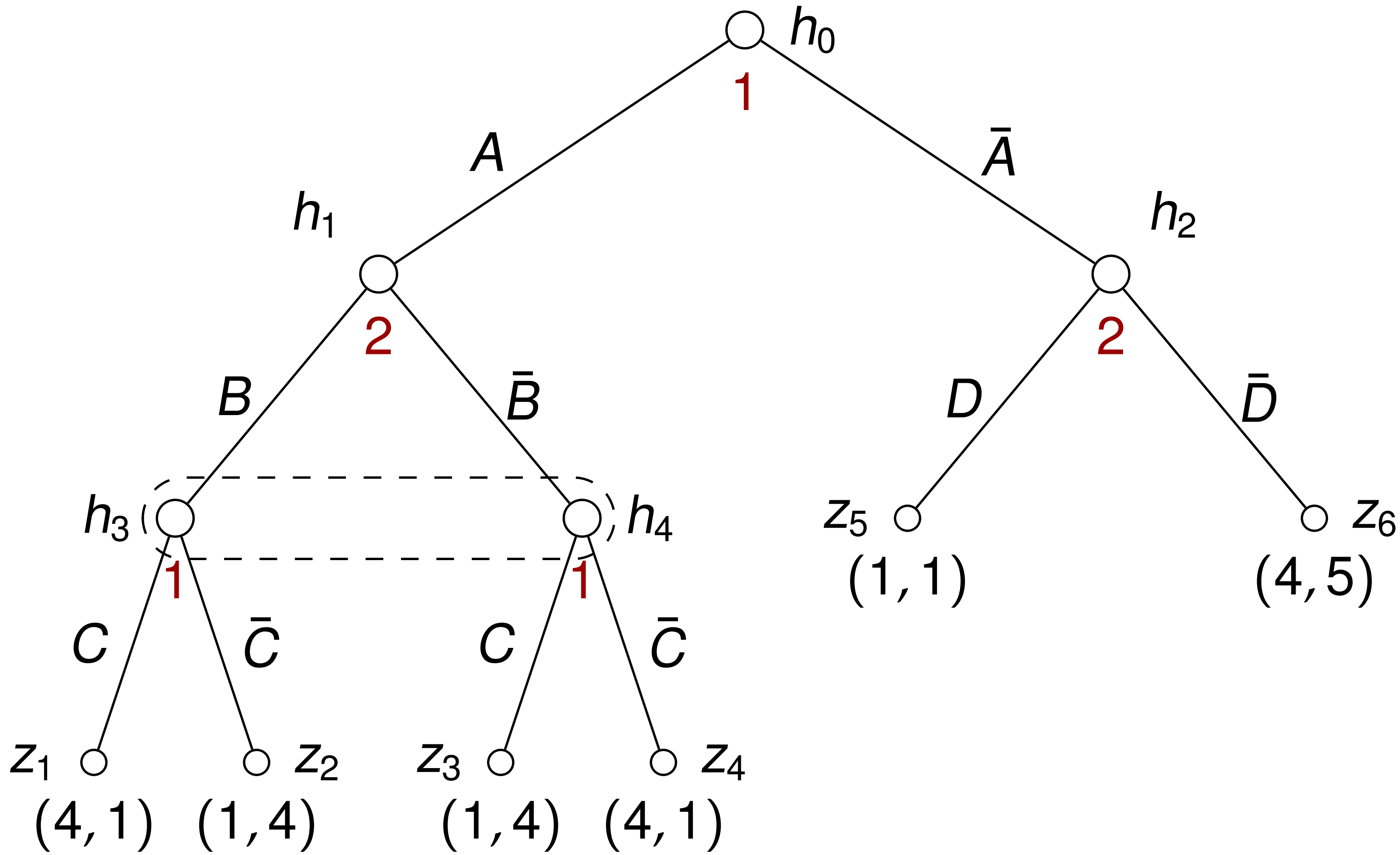



Note that $I_1 = \{I_{1,1}\}$ where $I_{1,1} = \{h_0, h_3\}$

and that $I_2 = \{I_{2,1}\}$ where $I_{2,1} = \{h_1, h_2\}$

What pure strategies are in this example?

# SPE with Imperfect Information




What we designate as subgames to allow the backward induction?

Only subtrees rooted in $h_1$, $h_2$, and $h_0$ (together with all subtrees rooted in terminal nodes)

Note that subtrees rooted in $h_3$ and $h_4$ cannot be considered as "independent" subgames because their individual solutions cannot be combined to a single best response in the information set $\{h_3, h_4\}$.

# SPE with Imperfect Information

Let $G_{imp} = (G_{perf}, I)$ be an imperfect-information extensive-form game where $G_{perf} = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$ is the underlying perfect-information extensive-form game.

Let us denote by $H_{single}$ the set of all $h \in H$ such that $I_{\rho(h),j}$ containing $h$ satisfies $I_{\rho(h),j} = \{h\}$.
I.e. $h \in H_{single}$ iff $h$ is a "perfect-information" node in which player $\rho(h)$ knows precisely the node $h$.

**Definition 66**
For every $h \in H_{single}$ we define a subgame $G^h_{imp}$ to be the imperfect information game $(G^h_{perf}, I^h)$ where $I^h$ is the restriction of $I$ to $H^h$.
Note that as subgames we consider only subtrees rooted in "perfect-information" nodes, that is nodes whose corresponding information set is a singleton.

**Definition 67**
A strategy profile $s \in S$ is a subgame perfect equilibrium (SPE) if $s^h$ is a Nash equilibrium in every subgame $G^h_{imp}$ of $G_{imp}$ (here $h \in H_{single}$).

# Backward Induction with Imperfect Info

The backward induction generalizes to imperfect-information extensive-form games along the following lines:

1. As in the perfect-information case, the goal is to label each node $h \in H_{single} \cup Z$ with a SPE $s^h$ and a vector of payoffs $u(h) = (u_1(h), \ldots, u_n(h))$ for individual players according to $s^h$.
2. Starting with terminal nodes, the labeling proceeds bottom up. Terminal nodes are labeled similarly as in the perfect-inf. case.
3. Consider $h \in H_{single}$, let $K$ be the set of all $h' \in \left(H_{single} \cup Z\right) \smallsetminus \{h\}$ that are $h$'s closest descendants out of $H_{single} \cup Z$.

   I.e., $h' \in K$ iff $h' \neq h$ is reachable from $h$ and the unique path from $h$ to $h'$ visits only nodes of $\mathcal{H} \smallsetminus H_{single}$ (except the first and the last node).

   For every $h' \in K$ we have already computed a SPE $s^{h'}$ in $G^{h'}_{imp}$ and the vector of corresponding payoffs $u(h')$.
4. Now consider all nodes of $K$ as terminal nodes where each $h' \in K$ has payoffs $u(h')$. This gives a new game in which we compute an equilibrium $\bar{s}^h$ together with the vector $u(h)$.

   The equilibrium $s^h$ is then obtained by "concatenating" $\bar{s}^h$ with all $s^{h'}$, here $h' \in K$, in the subgames $G^{h'}_{imp}$ of $G^h_{imp}$.

# Mutually Assured Destruction

Analysis of Cuban missile crisis of 1962
(as described in *Games for Business and Economics* by R. Gardner)

- The crisis started with United States' discovery of Soviet nuclear missiles in Cuba.
- The USSR then backed down, agreeing to remove the missiles from Cuba, which suggests that US had a credible threat "if you don't back off we both pay dearly".

**Question:** Could this indeed be a credible threat?

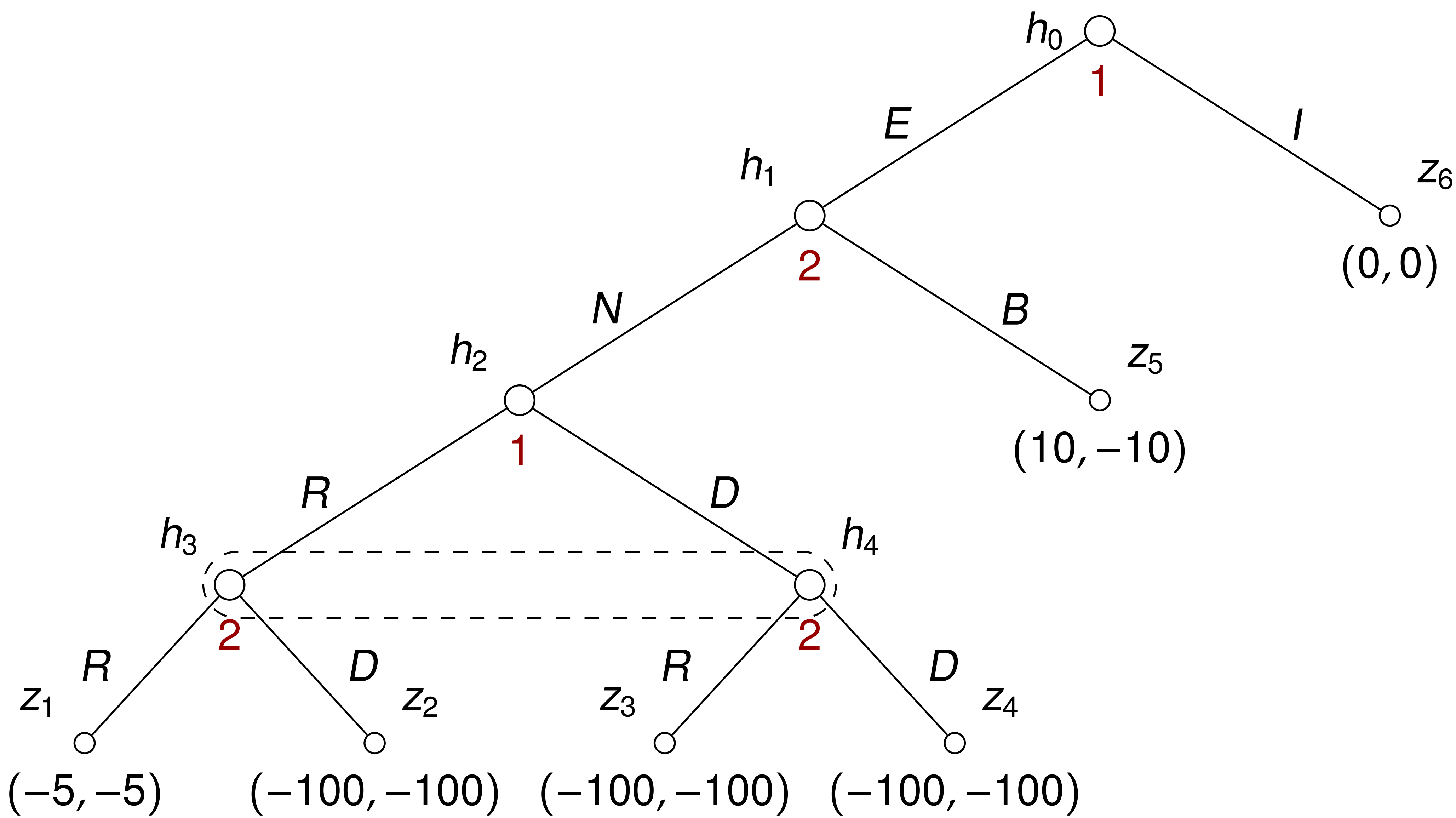
# Mutually Assured Destruction (Cont.)

Model as an extensive-form game:

- First, player 1 (US) chooses to either ignore the incident ($I$), resulting in maintenance of status quo (payoffs $(0, 0)$), or escalate the situation ($E$).
- Following escalation by player 1, player 2 can back down ($B$), causing it to lose face (payoffs $(10, -10)$), or it can choose to proceed to a nuclear confrontation ($N$).
- Upon this choice, the players play a simultaneous-move game in which they can either retreat ($R$), or choose doomsday ($D$).
    - If both retreat, the payoffs are $(-5, -5)$, a small loss due to a mobilization process.
    - If either of them chooses doomsday, then the world destructs and payoffs are $(-100, -100)$.

Find SPE in pure strategies.

# Mutually Assured Destruction (Cont.)




Solve $G^{h_2}_{imp}$ (a strategic-form game). Then $G^{h_1}_{imp}$ by solving a game rooted in $h_1$ with terminal nodes $h_2, z_5$ (payoffs in $h_2$ correspond to an equilibrium in $G^{h_2}_{imp}$). Finally solve $G_{imp}$ by solving a game rooted in $h_0$ with terminal nodes $h_1, z_6$ (payoffs in $h_1$ have been computed in the previous step).