$f: \mathbb{R} \to \mathbb{R}$

tečna v x_0: $y - y_0 = f'(x_0) \cdot (x - x_0)$

$y_0 = f(x_0)$

$f: \mathbb{R}^2 \to \mathbb{R}$

tečná rovina v $[x_0, y_0]$: $z - z_0 = \underbrace{f'(x_0, y_0)}_{(f'_x(x_0,y_0),\, f'_y(x_0,y_0))} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$

$$= \underbrace{f'_x(x_0, y_0) \cdot (x - x_0) + f'_y(x_0, y_0)(y - y_0)}_{df(x_0, y_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}}$$

Příklad

Určete rovnici tečné nadroviny ke grafu funkce v daném bodě:

a) $f(x,y) = x^2 + xy + 2y^2$, $\quad [x_0, y_0, z_0] = [1, 1, ?]$,

b) $f(x,y) = \operatorname{arctg} \frac{y}{x}$, $\quad [x_0, y_0, z_0] = [1, -1, ?]$.

a) $z_0 = f(x_0, y_0) = 4$

rce tečné roviny: $z - 4 = f'_x(1,1)\cdot(x-1) + f'_y(1,1)\cdot(y-1)$

$f'_x(x,y) = 2x + y \qquad f'_x(1,1) = 3$

$f'_y(x,y) = x + 4y \qquad f'_y(1,1) = 5$

$z - 4 = 3(x-1) + 5(y-1)$

$z = 3x + 5y - 4$

b) $f(x,y) = \operatorname{arctg} \frac{y}{x} \quad \sim [1, -1, ?]$

$z_0 = f(1,-1) = -\frac{\pi}{4}$

$f'_x(1,-1) = \frac{1}{2} \qquad f'_y(1,-1) = \frac{1}{2}$

$z + \frac{\pi}{4} = +\frac{1}{2}(x-1) + \frac{1}{2}(y+1)$

$x + y - 2z = -\frac{\pi}{2}.$

$$\left(\frac{\partial}{\partial x_j} \circ \frac{\partial}{\partial x_i}\right) f = \frac{\partial^2}{\partial x_i \partial x_j} f = \frac{\partial^2 f}{\partial x_i \partial x_j} = f''_{x_i x_j}$$

$$= f_{x_i x_j}$$

$Hf(x_0, y_0)$

$$(x - x_0, y - y_0) \cdot \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} =$$

$\boxed{\text{diferencer } (x - x_0, y - y_0)}$

$$= f''_{xx}(x_0, y_0) \cdot (x - x_0)^2 + 2 f''_{xy}(x_0, y_0) \cdot (x - x_0)(y - y_0)$$
$$+ f''_{yy}(x_0, y_0) \cdot (y - y_0)^2$$

$\frac{\partial f}{\partial x} = e^{x^3+y} \cdot 3x^2$, $\frac{\partial f}{\partial y} = e^{x^3+y}$, $\frac{\partial^2 f}{\partial x^2} = \left(e^{x^3+y} \cdot 3x^2\right)'_x$

$e^{x^3+y} \cdot (3x^2 \cdot 3x^2 + 6x)$, $\frac{\partial^2 f}{\partial xy} = e^{x^3+y} \cdot 3x^2$, $\frac{\partial^2 f}{\partial y^2} = e^{x^3+y}$.

$$T_2(0+\xi, 0+\eta) = f(0,0) + \frac{1}{1!} \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\binom{\xi}{\eta}$$

$$+ \frac{1}{2!}\left[(\xi,\eta) \cdot \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \cdot \binom{\xi}{\eta}\right] =$$

$$= 1 + \frac{1}{1!} \cdot (0, 1) \cdot (\xi, \eta) + \frac{1}{2!}\left((\xi,\eta) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\binom{\xi}{\eta}\right)$$

$$= 1 + \eta + \frac{1}{2}\eta^2$$

Příklad

Určete Taylorův polynom 2. stupně se středem v daném bodě:

a) $\ln\sqrt{x^2+y^2}$, $[x_0, y_0] = [1, 1]$,

b) $x^{\frac{y}{z}}$, $[x_0, y_0, z_0] = [1, 1, 1]$.

SAMI !

ad b) $f(x,y,z) = x^{\frac{y}{z}}$

$f'_x = x^{\frac{y}{z}-1}$

$f'_y = \left(e^{\frac{y}{z}\cdot\ln x}\right)'_y =$ [ad $f(x) = a^x$]

$= x^{\frac{y}{z}} \cdot \frac{\ln x}{z}$

f'_z obdobně ...

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Příklad

Pomocí Taylorova polynomu 2. stupně přibližně vypočtěte

a) $\sin 29° \operatorname{tg} 46°$, $\quad \xi = x - x_0 = 29° - 30° = -\frac{\pi}{180}, \; \eta = \frac{\pi}{180}$

b) $\ln(x^2 + y^2 + 1)$ v bodě $[1{,}1; 1{,}2]$.

a) $f(x,y) = \sin x \cdot \operatorname{tg} y \quad \sim \left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

$f'_x(x,y) = \cos x \cdot \operatorname{tg} y \qquad f'_x\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2} \cdot 1 = \frac{\sqrt{3}}{2}$

$f'_y(x,y) = \sin x \cdot \left(\frac{\sin y}{\cos y}\right)'_y = \sin x \cdot \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = \sin x \cdot \frac{1}{\cos^2 y}$

$f'_y\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} = 1$

$f''_{xx}(x,y) = -\sin x \cdot \operatorname{tg} y \qquad f''_{xx}\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = -\frac{1}{2}$

$f''_{xy}(x,y) = \cos x \cdot \frac{1}{\cos^2 y} \qquad f''_{xy}\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2} \cdot 2 = \sqrt{3}$

$f''_{yy}(x,y) = \sin x \left[(\cos y)^{-2}\right]'_y = \sin x \cdot (-2) \cdot (\cos y)^{-3} \cdot (-\sin y) = 2\frac{\sin x \cdot \sin y}{\cos^3 y}$

$f''_{yy}\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = 2 \cdot \frac{\frac{1}{2} \cdot \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2} \cdot \frac{1}{2}} = 2$

$$\sin 29° \operatorname{tg} 46° = \underbrace{\sin \frac{\pi}{6} \operatorname{tg} \frac{\pi}{4}}_{f(x_0,y_0)} + \frac{1}{1!} \cdot \left(\frac{\sqrt{3}}{2}, 1\right) \cdot (\xi, \eta)$$

$$+ \frac{1}{2!} \cdot (\xi, \eta) \underbrace{\begin{pmatrix} -\frac{1}{2} & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}}_{Hf(x_0,y_0)} \begin{pmatrix} \xi \\ \eta \end{pmatrix} =$$

$T_{2,\left[\frac{\pi}{6}, \frac{\pi}{4}\right]}(x, y)$, $\quad x = \frac{\pi}{6} + \xi, \; y = \frac{\pi}{4} + \eta$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \xi + 1 \cdot \eta + \frac{1}{2}\left(-\frac{1}{2}\xi^2 + 2\sqrt{3}\,\xi\eta + 2\eta^2\right)$$

$\doteq \underline{\underline{0{,}4973}}$

b) $\ln(x^2 + y^2 + 1)$ v bodě [1,1; 1,2].

$$f'_x(x;y) = \frac{2x}{x^2+y^2+1} \qquad f'_y(x,y) = \frac{2y}{x^2+y^2+1} \qquad f'_x|(1,1) = \frac{2}{3}$$

$$f'_y|(1,1) = \frac{2}{3}$$

$$f''_{xx}(x,y) = \frac{2(y^2 - x^2 + 1)}{(x^2+y^2+1)^2} \qquad f''_{yy} \text{ sym.}$$

$$f''_{xy}(x,y) = -\frac{4xy}{(x^2+y^2+1)^2} \Rightarrow Hf(1,1) = \begin{pmatrix} \frac{2}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{2}{9} \end{pmatrix}$$

$$T_{2,[1,1]}(x;y) = f(1,1) + \left(f'_x(1,1), f'_y(1,1)\right) \cdot (x-1, y-1)$$

$$+ \frac{1}{2!} \cdot \frac{1}{9}\left(2(x-1)^2 + 2 \cdot (-4) \cdot (x-1)(y-1) + 2(y-1)^2\right) \doteq$$

$$\doteq \ln 3 + 0{,}2 - \frac{1}{3}\, 0{,}01.$$

$$(x,y)\underbrace{\begin{pmatrix} a & b \\ b & d\end{pmatrix}}_{\text{Matice kvadratické formy}}\begin{pmatrix}x\\ y\end{pmatrix} = ax^2+2bxy+dy^2$$

$(x,y)\mapsto ax^2+2bxy+dy^2$

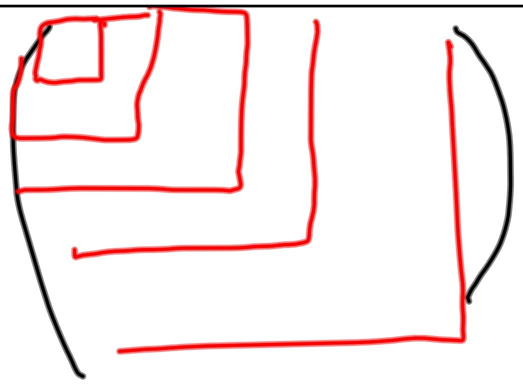
Př: $(x,y)\mapsto x^2+y^2$ pozitivně def.

$(x,y)\mapsto -x^2-y^2$ neg. def.

$(x,y)\mapsto x^2$ pozit. semidef.

$(0,1)\mapsto 0$

$(x,y)\mapsto x^2-y^2$ indefinitní



hlavní minory

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ x^2+y^2, vl. h. $1, 1$

hl. minory $1, 1$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $-x^2-y^2$ vl. h. $-1, -1$

neg. def., hl. minory $-1, 1$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $-x^2+y^2$ indef.,

hl. min. $-1, -1$

	$S_1 = [0,0]$	$S_2 = [0,1]$	$S_3 = \left[\frac{1}{3}, \frac{1}{3}\right]$	$S_4 = [1,0]$
f''_{xx}	0	2	$\frac{2}{3}$	0
f''_{xy}	-1	1	$\frac{1}{3}$	1
f''_{yy}	0	0	$\frac{2}{3}$	2
Hessián	-1	-1	$\frac{1}{3}$	-1
Závěr	sedlo	sedlo	minimum	sedlo

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

indef.

$\frac{2}{3}, \frac{1}{3}$ poz. def.
lok. min.