

$$1) \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & -1 & 1 & 0 \\ -2 & 1 & 1 & -1 \\ 2 & -1 & -1 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix}}_A \cdot \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} \quad \text{I.}$$

$$\det(A - \lambda I) = \lambda^2(\lambda^2 - 1) = 0 \dots \quad \lambda_{1,2} = 0, \quad \lambda_3 = 1, \quad \lambda_4 = -1$$

$$\underline{\lambda_3 = 1}$$

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 0 & 1 & -1 \\ 2 & -1 & -2 & 1 \\ 0 & 2 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0 \dots \quad -a - b$$

2

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ -2 & 0 & 1 & -1 \\ 0 & 2 & -1 & -1 \end{pmatrix} \approx \begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -3 & -1 \\ -2 & 0 & 1 & -1 \end{pmatrix} \begin{matrix} \cdot (-2) \\ \cdot (2) \end{matrix} \approx \begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & -3 & -1 \end{pmatrix}$$

$$\approx \begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \dots$$

$$-a - b + c = 0$$

$$-b - c = 0 \Rightarrow$$

$$-3c - d = 0$$

$$\underline{d = 3c}$$

$$\underline{b = -c}$$

$$\underline{a = c - b = 2c}$$

$$\begin{pmatrix} 2c \\ -c \\ c \\ 3c \end{pmatrix} \Rightarrow c = 1 \dots \quad \boxed{\begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} \cdot e^t}$$

$$\underline{\lambda_4 = -1}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ -2 & 2 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 0 & 2 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \approx \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -1 \\ 2 & -1 & 0 & 1 \\ 0 & 2 & -1 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 3 & -1 \\ 2 & -1 & 0 & 1 \end{pmatrix} \begin{matrix} (2) \\ \\ \\ \end{matrix}$$

II.

$$\approx \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{pmatrix} \xrightarrow{+2} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -3 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

...  $a - b + c = 0$

$2b - c + d = 0$  ...  $d = 3c$

$3c - d = 0$  ...  $2b = c - d = c - 3c = -2c$  ...  $b = -c$

$a = b - c = -c - c = -2c$

$$\begin{pmatrix} -2c \\ -c \\ c \\ 3c \end{pmatrix} \rightarrow c = 1 \rightarrow \boxed{\begin{pmatrix} -2 \\ -1 \\ 1 \\ 3 \end{pmatrix} e^{-t}}$$

$\lambda_{1,2} = 0$   $\rightarrow 4 - 2 = 4 - 2 = \underline{2}$

$$A - \lambda I = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -2 & 1 & 1 & -1 \\ 2 & -1 & -1 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix} \dots (A - \lambda I)^0 = I_4 \dots r_0 = 4$$

$(A - \lambda I)^1 = A - \lambda I \dots r_1 = 3$

$$(A - \lambda I)^2 = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -2 & 1 & 1 & -1 \\ 2 & -1 & -1 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 1 & 0 \\ -2 & 1 & 1 & -1 \\ 2 & -1 & -1 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & -2 & -2 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 3 & 3 & -3 \end{pmatrix}$$

$\rightarrow r_2 = 2$

$k: 4 > 3 > 2 \rightarrow v_1 = v_1 - v_0 = 1 - 0 = 1$

$v: 0 < 1 < 2 \rightarrow v_2 = v_2 - v_1 = 2 - 1 = 1$

$v_{1,1}$
$v_{2,1}$

III.

$$v_{21}: (A - \lambda \cdot I)^2 \cdot v_{21} = 0 \rightarrow \begin{pmatrix} 4 & -2 & -2 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 3 & 3 & -3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

$$v_{11}: v_{11} = (A - \lambda \cdot I) v_{21}$$

$$\begin{pmatrix} 2 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{matrix} 2a - b - c + d = 0 \\ c = 0 \\ 2a - b + d = 0 \end{matrix}$$

$$\begin{pmatrix} a \\ b \\ 0 \\ b - 2a \end{pmatrix} = v_{21}$$

$$v_{11} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -2 & 1 & 1 & -1 \\ 2 & -1 & -1 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ 0 \\ b - 2a \end{pmatrix} = \begin{pmatrix} -b \\ 0 \\ 0 \\ 2b \end{pmatrix} \Rightarrow \begin{matrix} b = 1 \\ a = 1 \end{matrix}$$

$$v_{21} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad ; \quad v_{11} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ +2 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ +2 \end{pmatrix} \cdot e^{0 \cdot t} = \boxed{\begin{pmatrix} -1 \\ 0 \\ 0 \\ +2 \end{pmatrix}} \quad ; \quad \left[ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ +2 \end{pmatrix} \right] \cdot e^{0 \cdot t} =$$

$$= \boxed{\begin{pmatrix} 1-t \\ 1 \\ 0 \\ -1+2t \end{pmatrix}}$$

$$Y(t) = \begin{pmatrix} 2e^t & -2e^{-t} & -1 & 1-t \\ -e^t & -e^{-t} & 0 & 1 \\ e^t & e^{-t} & 0 & 0 \\ 3e^t & 3e^{-t} & 2 & 2t-1 \end{pmatrix}$$