MA010 Tutorial 4—Solution to Problem 4

Frédéric Dupuis

Problem

A *maximum independent set* is a set of vertices such that no two of them are joined by an edge. A *covering of vertices by edges* is a set of edges such that every vertex is an endpoint of at least one of these edges. Prove that if G is a bipartite graph with every vertex having degree at least 1, then the size of the largest independent set is equal to the size of the smallest covering of vertices by edges.

Hint: Think of the relationship with König's theorem (i.e. max matching = min vertex cover in bipartite graphs). What is the complement of a vertex cover?

Solution

Denote the graph by $G = (V, E)$, let $c(G)$ be the size of the minimum edge cover, and let $i(G)$ be the size of the maximum independent set. We prove the statement in two steps: first, we show that $i(G) \leqslant c(G)$, and then we show that there exists an edge cover of the same size as the maximum independent set.

To show that $i(G) \leqslant c(G)$, suppose that we have an edge cover $E_c \subseteq E$ and an independent set $I \subseteq V$ with $|I| > |E_c|$. Since E_c is an edge cover, every vertex $v \in I$ must be adjacent to at least one edge in E_c, and since $|I| > |E_c|$, there must be at least one edge in E_c adjacent to two vertices in I. This contradicts the fact that I is an independent set.

Now, we show that there must exist an edge cover E_c with $|E_c| = i(G)$. To do this, we first observe that the complement of a vertex cover is an independent set, and vice-versa. Now, König's theorem tells us that in a bipartite graph, the size $m(G)$ of a maximum matching is equal to the size $v(G)$ of a minimum vertex cover.

We now construct the edge cover E_c as follows. First, we find a maximum matching E_m in G, and derive from it a minimum vertex cover $V_c \subseteq V$, such that each edge in E_m is adjacent to exactly one vertex in V_c (and therefore to one vertex in the maximum independent set $V \in V_c$). We then add to E_c all the edges in the matching E_m. This means that E_c now automatically covers all vertices in V_c (as well as an equal number of vertices from $V \backslash V_c$). Now, for each vertex $v \in V \backslash V_c$ that remains uncovered, we add one edge. Since $V \backslash V_c$ is an independent set, we need exactly one edge per uncovered vertex. Now we are done: E_c is clearly an edge cover, and there is a one-to-one mapping between edges in E_c and vertices in $V \backslash V_c$, and therefore $|E_c| = |V \backslash V_c|$.