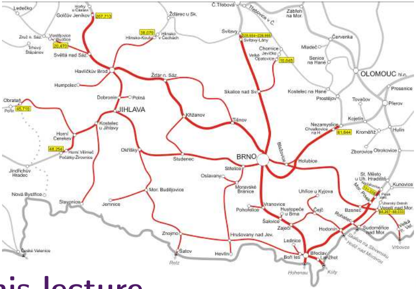
3 Graph Distance and Path Finding

In some other applications, graphs are used to model distances; e.g., as in road networks and in workflow diagrams. The basic task then is to find shortest paths or routes, and the optimal distance.



Brief outline of this lecture

- Distance in a graph, basic properties, BFS.
- Weighted distance in digraphs; the problem of negative cycles and Bellman–Ford's algorithm.
- Dijkstra's algorithm for the single-source shortest paths.
- A sketch of some advanced ideas in practical path planning.

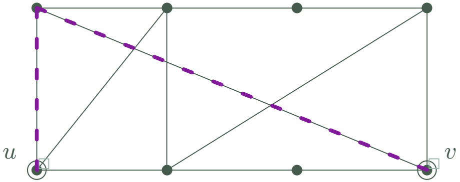
3.1 Unit Distance in Graphs

Recall that a *walk of length n* in a graph G is an alternating sequence of vertices and edges $(v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n)$ such that each e_i has the ends v_{i-1}, v_i.

Definition 3.1. ***The distance*** $d_G(u,v)$ between two vertices u, v of a graph G is defined as the length of a shortest walk between u and v in G.

If there is no walk between u, v, then we declare $d_G(u,v) = \infty$. □

Naturally, the distance between u, v equals *the least possible number of edges* travelled from u to v, and it is always achieved by a path, as shown in Lemma 2.6. Spec. $d_G(u,u) = 0$.

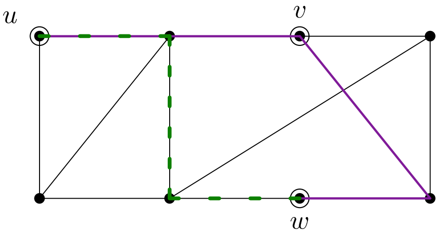


Remark: Distance can be analogously defined for digraphs, using directed walks or paths.

A more general view in Section 3.2 will consider also non-unit lengths of edges in G.

walk of length = procházka délky, distance = vzdálenost

Triangle inequality



Lemma 3.2. *The graph distance satisfies the triangle inequality:*

$$\forall u, v, w \in V(G) \ : \quad d_G(u, v) + d_G(v, w) \geq d_G(u, w) \,. \square$$

Proof. Easily; starting with a walk of length $d_G(u, v)$ from u to v, and appending a walk of length $d_G(v, w)$ from v to w, results in a walk of length $d_G(u, v) + d_G(v, w)$ from u to w. This is an upper bound on the distance from u to w. □ □

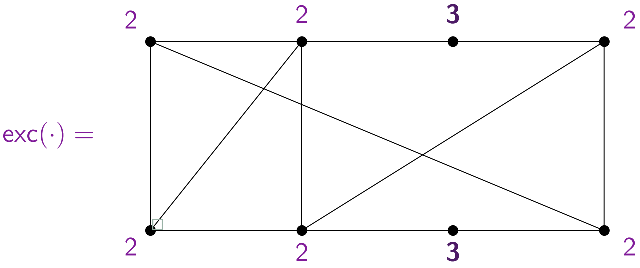
Fact: The distance in an undirected graph is symmetric, i.e. $d_G(u, v) = d_G(v, u)$.

triangle inequality = trojúhelníková nerovnost

Other related terms

Definition 3.3. Let G be a graph. We define, with resp. to G, the following notions:

- The *excentricity* of a vertex $\mathrm{exc}(v)$ is the largest distance from v to another vertex; $\mathrm{exc}(v) = \max_{x \in V(G)} d_G(v, x)$. □
- The *diameter* $\mathrm{diam}(G)$ of G is the largest excentricity over its vertices, and the *radius* $\mathrm{rad}(G)$ of G is the smallest excentricity over its vertices.



It always holds $\mathrm{diam}(G) \leq 2 \cdot \mathrm{rad}(G)$. □

- The *center* of G is the subset $U \subseteq V(G)$ of vertices such that their excentricity equals $\mathrm{rad}(G)$.

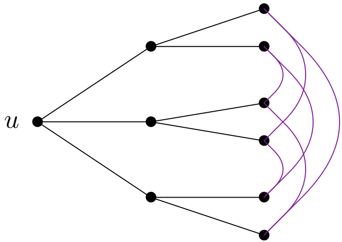
diameter = průměr, radius = poloměr
excentricity = excentricita

An excersise

Example 3.4. *What is the largest possible number of vertices a cubic (i.e., 3-regular) graph of radius 2 may have?* □

Let G be the graph. First of all, the definition of radius tells us that, for some vertex $u \in V(G)$, all the vertices of G are at distance ≤ 2 from u. □

Second, there can be ≤ 10 such vertices by the degree-3 condition:



□

And third, we are able (or lucky?) to fill in the remaining six edges (in order to get all the degrees equal to 3) as in the picture. Hence, 10 vertices is possible, and this is the answer. □ □

Remark: Note, moreover, that we have actually constructed a graph of *diameter* 2, which is a stronger requirement than *radius* 2.

Simple Computation of Distance (BFS)

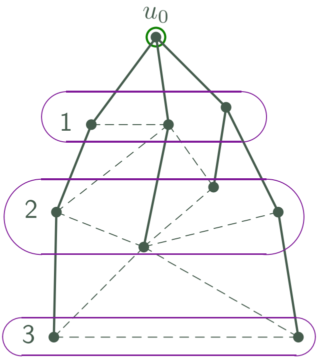
Computing the (unit) distance from a given vertex u_0 to any other vertex of a graph is a matter of an extremely simple algorithm, based on BFS:

Algorithm 3.5. ***Computing all distances from a starting vertex $u_0 \in V(G)$.*** □
For a given graph (or digraph) G and any $u_0 \in V(G)$, we run Algorithm 2.1 with the implementation of `PROCESS(v;e)` *as follows (and with void* `PROCESS(e)`*):*

`U` *as a fifo queue (BFS), and*

initialize `dist[u0,v]` $\leftarrow \infty$, *for all* $v \in V(G)$;
`dist[u0,u0]` $\leftarrow$ `0;`

```
...
PROCESS(v;e) {
    u ← the starting vertex of 'e = uv';
    dist[u0,v] ← dist[u0,u]+1;
}
```



** jednoduchý výpočet vzdálenosti přes BFS **

BFS distance – the proof

Theorem 3.6. *Let u_0, v, w be vertices of a connected graph G such that $d_G(u_0, v) < d_G(u_0, w)$. Then the breadth-first search algorithm on G, starting from u_0, discovers the vertex v before w.* □

Proof. We apply induction on the distance $d_G(u_0, v)$: If $d_G(u_0, v) = 0$, i.e. $u_0 = v$, then it is trivial that v is found first. So let $d_G(u_0, v) = d > 0$ and v' be a neighbour of v closer to u_0, which means $d_G(u_0, v') = d - 1$. Analogously choose w' a neighbour of w closer to u_0. Then

$$d_G(u_0, w') = d_G(u_0, w) - 1 > d_G(u_0, v) - 1 = d_G(u_0, v')\,,$$ □

and so v' has been found before w' by the inductive assumption. Hence v' has been stored into U before w', and (cf. *FIFO*) the neighbours of v' (v among them, but not w) are discovered before the neighbours of w' (which include w). □ □

Corollary 3.7. *The search tree of the BFS Algorithm 2.1 on G determines the distances from $u_0 \in V(G)$ to all vertices of G.*

Hence, Alg. 3.5 is correct, meaning that $\texttt{dist}(u_0, v) = d_G(u_0, v)$ *for all* $v \in V(G)$.

3.2 Weighted Distance in Digraphs

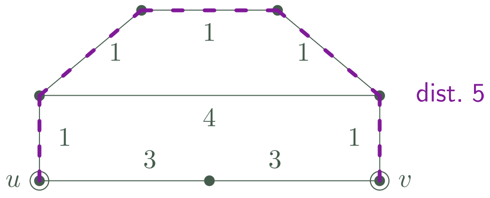
Recall (Section 2.3): A *weighted (di)graph* is a pair of a (di)graph G together with a weighting w of the edges by real numbers $w : E(G) \to \boldsymbol{R}$ (edge lengths in this case).
A *positively weighted (di)graph* (G, w) is such that $w(e) > 0$ for all edges e. □

Definition 3.8. ***Weighted distance (length)*** in a weighted (di)graph (G, w).
The *length* of a weighted (dir.) walk $S = (v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n)$ in G is the sum

$$d_G^w(S) = w(e_1) + w(e_2) + \cdots + w(e_n)\,. \square$$

The *weighted distance* in (G, w) from a vertex u to a vertex v is

$$d_G^w(u, v) = \min\{d_G^w(S) : S \text{ is a (directed) walk from } u \text{ to } v\}\,. \square$$

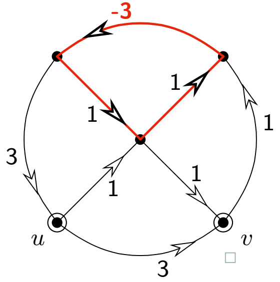


For undir. graphs G, the definition considers the symmetric orientation of the edges.

weighted distance (length) = vážená vzdálenost (délka)

Basic facts

- Weighted distance in a digraph (G, w) satisfies the triangle inequality. (The same statement and proof hold here as in Lemma 3.2.) □
- Ordinary graph distance is obtained for weights (G, w_1) s.t. $w_1(e) = 1$ for all e.□
- If a weighted digraph (G, w) contains a cycle (a closed walk) of negative length, then the distance between a pair of vertices in G may not be defined ("$-\infty$"):



Proposition 3.9. *If (G, w) is a weighted digraph containing no cycles of negative length (and hence no such closed walks), then* □

- *the weighted distance in (G, w) is always well defined, and* □
- *the weighted distance is achieved by a directed path in G.*

Negative or positive weights?

- By the previous facts, negative-length edges may cause huge problems with graph distance. So, why to consider them at all?

  (Do they make sense, anyway?) □

- For undirected graphs, the negative-length problem seems fatal, and hence we consider only positively weighted undirected graphs.

  For digraphs, though, negative-length edges might be useful to consider, as long as there is no cycle of negative length (Prop. 3.9). E.g., for DAGs.

Bellman–Ford Algorithm

Definition: A cycle of negative length in a weighted digraph is called a *negative cycle*.

Algorithm 3.10. ***Computing the distance or detecting a negative cycle.***
For a given weighted digraph (G, w), and a starting vertex $u_0 \in V(G)$, the task is to compute the distance $dist[u_0, v] = d_G^w(u_0, v)$ from u_0 to any vertex $v \in V(G)$.

```
initialize dist[u0,v] ← ∞,   for all v ∈ V(G);
dist[u0,u0] ← 0;
repeat |V(G)| − 1 times {
    foreach (e = tv ∈ E(G)) {
        dist[u0,v] ← min(dist[u0,v], dist[u0,t] + w(e));   (*)
    }
}
foreach (e = tv ∈ E(G)) {
    if (dist[u0,v] > dist[u0,t]+w(e))
        output 'Error; a negative cycle exists in (G,w).'
}
output 'Distances from u0 in dist[u0,·].'
```

(One can also easily store the predecessors for the computed distances on line (*). . .)

negative cycle = záporný cyklus, predecessor = předchůdce

Proof of the Bellman–Ford algorithm

Proof. To claim that $dist[u_0, v] = d^w_G(u_0, v)$ if there is no negative cycle in (G, w), and that a negative cycle is detected otherwise, we prove the following three steps.

1. At every step of Algorithm 3.10, it is $dist[u_0, v] \geq d^w_G(u_0, v)$: □

   This holds at the beginning, and follows trivially by induction on the number of elementary steps '$\texttt{dist[u}_0\texttt{,v]} \leftarrow \texttt{min(dist[u}_0\texttt{,v],dist[u}_0\texttt{,t]+}w\texttt{(e))}$'.□

2. Assume there is no negative dir. cycle in (G, w). Let (cf. Prop. 3.9) $V_k \subseteq V(G)$ be the subset of vertices v for which $d^w_G(u_0, v)$ is achieved by a dir. u_0-v path with $\leq k$ edges. Then, after iteration no. k of '$\texttt{foreach (e=uv} \in E(G)\texttt{)}$', the value of $\texttt{dist[u}_0\texttt{,v]}$ equals $d^w_G(u_0, v)$ for all $v \in V_k$: □

   Again, this trivially holds for $k = 0$ and follows easily by induction. □

3. Let $C \subseteq G$ be any directed cycle. If no negative cycle is reported at the end of Alg. 3.10, i.e. '$\texttt{dist[u}_0\texttt{,v]} \not> \texttt{dist[u}_0\texttt{,u]+}w\texttt{(e)}$' for all $e = uv \in E(C)$ in the last phase, then C is not a negative cycle in (G, w): □

   We have $dist(u_0, v) - dist(u_0, t) \leq w(e)$ and summing these over all $e \in E(C)$ we get $0 \leq \sum_{e \in E(C)} w(e)$. Consequently, negative cycles in (G, w) are detected in the algorithm (but only detected, they cannot be easily constructed).

□

3.3 Positive-length Shortest Paths

In contrast to previous Algorithm 3.10, shortest paths may be computed much faster when all the edge lengths are positive (which is true, e.g., in practical *routing problems*).

For the typical, so-called *single-source positive-length shortest paths problem*, a nearly optimal algorithm is the following traditional one.

Dijkstra's algorithm:

- For a given positively weighted digraph (G, w), and an arbitrary starting vertex $u_0 \in V(G)$, the algorithms computes $dist[u_0, v]$ for all $v \in V(G)$. □
- In the graph-search scheme of Algorithm 2.1, one simply implements
  - '*choose* `(e,u)` $\in$ `U`' by picking (e, u), $e = tu$, from U such that $dist(u_0, t) + w(tu)$ is minimized, □
  - '`PROCESS(u;e=tu)`' as `dist[`u_0`,u]` $\leftarrow$ `dist[`u_0`,t]+`w`(tu)`, □
  - '`PROCESS(e)`' as void, and
  - the search tree T then stores shortest paths from u_0. □
- This algorithm works in the same way for undirected as for directed graphs.

** nejkratší cesty z jednoho zdroje při kladných délkách **

A self-contained exposition of Dijkstra's algorithm is quite simple:

Algorithm 3.11. ***Dijkstra's for single-source shortest paths.***
For a positively weighted digraph (G,w), and a vertex $u_0 \in V(G)$, compute shortest paths $predec[\cdot]$ and distances $dist[u_0,\cdot]$ in (G,w) from the source u_0 to all of G.

```
initialize dist[u0,v] ← ∞,   for all v ∈ V(G);
dist[u0,u0] ← 0;
U ← {u0}; □
while (U ≠ ∅) {
    choose u∈U minimizing dist[u0,u];
    foreach (edge f starting in u) {
        v ← the opposite vertex of 'f = uv';
        if (dist[u0,u]+w(uv)<dist[u0,v]) {
            U ← U∪{v};
            predec[v] ← u;
            dist[u0,v] ← dist[u0,u]+w(uv);
        }
    }
    U ← U\{u};
}
output 'distances in dist[.], predecessors of shortest paths in predec[]';
```

** Dijkstrův algorithmus pro nejkratší cesty **

Proposition 3.12. *If the stack U is implemented as a minimum heap, then the number of steps performed by Algorithm 3.11 is $O\big(|E(G)| + |V(G)| \cdot \log |V(G)|\big)$.* □

A vertex $u \in V(G)$ is called "*relaxed*" after it is removed in '$\mathtt{U} \leftarrow \mathtt{U} \setminus \{\mathtt{u}\}$' above.

Theorem 3.13. *Every iteration of Algorithm 3.11 maintains an invariant that*

- $\mathtt{dist[u_0,v]}$ *is the length of a shortest path from* $\mathtt{u_0}$ *to* $\mathtt{v}$ *using only those internal vertices which are relaxed, and such a shortest path is stored in* `predec[.]`. □

Consequently, all the distances and shortest paths to reachable vertices are correct.

Proof: Briefly using *mathematical induction*:

- In the first iteration of '$\mathtt{while}\ (\mathtt{U} \neq \emptyset)$', u_0 is chosen and the straight distances (edge lengths) to its neighbours are stored. □
- Subsequently, for every chosen vertex u in '$\mathtt{u} \in \mathtt{U}$ *minimizing* $\mathtt{dist[u_0,u]}$', the current value of $\mathtt{dist[u_0,u]}$ is optimal since no negative edges exist in (G, w) (and so every possible detour via non-relaxed vertices would only be longer). □

  Then, all working distances and the shortest-paths record are properly updated (wrt. u) while "relaxing" u:

```
if (dist[u0,u]+w(uv)<dist[u0,v]) {
    predec[v] ← u;  dist[u0,v] ← dist[u0,u]+w(uv);
```

□

Bidirectional Dijkstra's algorithm

In some settings, the following improved variant may be significantly more efficient in the single-pair shortest path problem in a digraph (G, w): □

- To find a shortest u_0–v_0 path, run *two instances* of Algorithm 3.11 concurently:
  - $\mathcal{A}$ searches shortest paths from u_0 in (G, w), as usual, and □
  - $\mathcal{A}^{\leftarrow}$ searches shortest paths from v_0 in $(G^{\leftarrow}, w)$ where $G^{\leftarrow}$ results from G by reversing all edges; $e \in E(G)$ to $e^{\leftarrow} \in E(G^{\leftarrow})$ such that $w(e^{\leftarrow}) = w(e)$.□
- $\mathcal{A}$ and $\mathcal{A}^{\leftarrow}$ may simply alternate their iterations, or better;
  - minima $u \in U$ and $u' \in U^{\leftarrow}$ are chosen concurently, and the instance achieving smaller value among $dist(u_0, u)$ and $dist^{\leftarrow}(v_0, u')$ is run. □
- Termination condition;
  the whole algorithm stops when the search subtrees T and $T^{\leftarrow}$ of $\mathcal{A}$ and $\mathcal{A}^{\leftarrow}$ meet each other. That is, whenever some vertex is relaxed in both $\mathcal{A}$ and $\mathcal{A}^{\leftarrow}$.□

  At this moment, a shortest u_0–v_0 path exists using only relaxed vertices.
  Though, this does not mean that every shortest u_0–v_0 has to pass through the intersection of the two search trees – one still has to loop over the vertices x to determine the minimum of $dist(u_0, x) + dist^{\leftarrow}(v_0, x)$.

** obousměrný Dijkstrův algorithmus **

All-pairs Shortest Distances

The last algorithm we are going to present in this section is extraordinarily simple, although rather slow since it has to compute all-pairs distances at once. □

Algorithm 3.14. ***Floyd–Warshall's algorithm for all-pairs distances***
For a positively weighted digraph (G, w), compute distances $dist[\cdot,\cdot]$ between all pairs of vertices of G.

```
initialize dist[u,v] ← ∞,   for all u,v ∈ V(G);
foreach (uv ∈ E(G))  dist[u,v] ← w(uv);
foreach (t ∈ V(G)) {
    foreach (u,v ∈ V(G)) {
        dist[u,v] ← min(dist[u,v],dist[u,t]+dist[t,v]);
    }
}
output 'The complete distance matrix of (G,w) in d[,]';
```

The number of steps of this algorithm is $O\big(|V(G)|^3\big)$, which is quite slow compared to repeated Dijkstra in the case of sparse graphs. □

Remark: Floyd–Warshall's algorithm has many shapes; it appears, e.g., in computation of the transitive closure and in the translation of a finite automaton to a regular expression. □

The algorithm is also related to *matrix multiplication*.

Algorithm 3.14 is based on the following beautifully simple dynamic-programming idea:

Computing all-pairs distances dynamically

- Given is a weighted (di)graph (G, w) on n vertices; $V(G) = \{t_0, t_1, \ldots, t_{n-1}\}$.□

  Let $dist^i(u, v)$ denote the length of a shortest u–v walk S in G such that all vertices of S except the ends u, v are from the subset $\{t_0, \ldots, t_{i-1}\}$. □

- For computing $dist^{i+1}$, the admissible walks are those as for $dist^i$ plus those walks passing through t_i ("u–t_i–v"). □

  Consequently,

$$dist^{i+1}(u, v) = \min\left(dist^i(u, v),\ dist^i(u, t_i) + dist^i(t_i, v)\right)\square$$

  and

$$dist[u, v] = dist^n(u, v).\square$$

- This algorithm works correctly also with negative edge lengths, as long as there is no negative cycle (same as Bellman–Ford):

Proposition 3.15. *Algorithm 3.14 correctly computes distances between all pairs of vertices in a weighted (di)graph (G, w), provided that there is no negative cycle.*

3.4 Some Advanced Ideas in Path Finding

Based on the above comparison of approaches, *Dijkstra's algorithm* seems to be the ultimate tool for practical path finding (or *route planning*) problems.

- Being quite fast and, actually, "almost optimal" for the shortest path problem in weighted graphs, □Dijkstra's algorithm turns out to be too slow for, e.g., practical route planning applications in navigation devices containing map data of tens or hundreds millions of edges. □
- So, what can be done better? □
- An answer lies in *preprocessing* of the graph:

  It is quite natural to assume that the graph (of a road network) is relatively stable, and hence it can be thoroughly preprocessed on powerful computers. □

  However, what of the preprocessing results can be stored? It is, say, completely unrealistic to store all the optimal routes in advance. . . □
- Two perhaps simplest practically usable approaches will be briefly sketched next.

First, an alternative to Dijkstra's alg. is the *Algorithm A^**, which uses a suitable *potential function* to direct the search "towards the goal". Whenever we have a good "sense of direction" (e.g. in a topo-map navigation), A^* can perform way much better!

Algorithm A^*

- In a basic setting, A^* re-implements Dijkstra with suitably modified edge costs on digraphs. □
- Let $p_v(x)$ be a potential function giving an arbitrary lower bound on the distance from x to the destination v (i.e., p_v is *admissible*).
  E.g., in a map navigation, $p_v(x)$ may be the Euclidean distance from x to v. □
- Each oriented edge xy of the weighted graph (G, w) gets a new cost

  $$\boldsymbol{w'(xy) := w(xy) + p_v(y) - p_v(x)}\,.$$

  The potential p_v is *consistent* when all $w'(xy) \geq 0$, i.e. $w(xy) \geq p_v(x) - p_v(y)$.
  The above Euclidean potential is always consistent. □
- The modif. length of any u–v walk S then is $d_G^{w'}(S) = d_G^w(S) + p_v(v) - p_v(u)$, which is a constant difference from $d_G^w(S)$. □
  Consequently, some S is optimal for the weighting w iff S is optimal for w'.

  Here the Euclidean potential "strongly prefers" edges in the destin. direction. Other (also preprocessed) potential functions are possible as well, though.

Second, ...

Idea of the "reach" parameter

- It is based on a natural observation that for long-distance route planning, vaste majority of edges of real-world road maps are basically "irrelevant". □

Definition: Let $S_{u,v}$ denote a shortest walk from u to v in weighted G. For $e \in E(S_{u,v})$ let $prefix(S_{u,v}, e)$, $suffix(S_{u,v}, e)$ denote the starting (ending) segment of $S_{u,v}$ up to (after) e. □ The *reach of an edge* $e \in E(G)$ is given as

$$reach_G(e) = \max \Big\{ \min\Big(d^w_G(prefix(S_{u,v}, e)), d^w_G(suffix(S_{u,v}, e)) \Big) : \forall u, v \in V(G) \wedge e \in E(S_{u,v}) \Big\}. \square$$

The reach of e mathematically quantifies (ir)relevance of e for route planning; the smaller $reach_G(e)$ is, the closer to the start or end of an optimal route e has to be. □

The immediate use of precomputed reach values is as follows:

- We must use the bidirectional variant of Dijkstra or A^*. □
- The line '`foreach` (*edge* `f` *starting in* `u`)' in Algorithm 3.11 (in each direction) now takes only those edges $f = uv$ such that $reach_G(f) \geq dist[u_0, u]$.

3.5 Appendix: An example run of Dijkstra's alg.

Example 3.15. *An illustration run of Dijkstra's Algorithm 3.11 from u to v in the following graph.*