

**Exercise 1** Prove that the following formulae are valid using Natural Deduction.

- (a)  $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$
- (b)  $\psi \rightarrow ((\varphi \wedge \psi) \vee \psi)$
- (c)  $(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$
- (d)  $\varphi \rightarrow \neg\neg\varphi$
- (e)  $((\varphi \wedge \psi) \vee \psi) \rightarrow \psi$
- (f)  $\neg\neg\varphi \rightarrow \varphi$
- (g)  $\varphi \vee \neg\varphi$
- (h)  $\neg(\neg\varphi \wedge \neg\psi) \rightarrow (\varphi \vee \psi)$  (This one is tricky.)
- (i)  $\varphi \rightarrow \exists x\varphi$
- (j)  $\forall x\varphi \rightarrow \varphi$
- (k)  $\forall xR(x, x) \rightarrow \forall x\exists yR(f(x), y)$
- (l)  $\exists x(\varphi \vee \psi) \rightarrow (\exists x\varphi \vee \exists x\psi)$
- (m)  $(\exists x\varphi \vee \exists x\psi) \rightarrow \exists x(\varphi \vee \psi)$
- (n)  $\forall x\varphi \wedge \forall x\psi \rightarrow \forall x(\varphi \wedge \psi)$
- (o)  $\forall x(\varphi \wedge \psi) \rightarrow \forall x\varphi \wedge \forall x\psi$
- (p)  $\forall x\forall y[\varphi(x) \leftrightarrow \varphi(y)] \wedge \exists x\varphi(x) \rightarrow \forall x\varphi(x)$

**Exercise 2** Prove that the formulae from Exercise 1 are valid using tableau proofs.

**Exercise 3** Find all consistent sets for the following sets of rules.

- (a)  $\frac{}{\alpha} \quad \frac{\alpha:\beta}{\delta} \quad \frac{\alpha:\gamma}{\delta}$
- (b)  $\frac{}{\alpha} \quad \frac{\alpha:\beta\gamma}{\beta}$
- (c)  $\frac{}{\alpha} \quad \frac{\alpha\beta}{\gamma} \quad \frac{\alpha:\gamma}{\beta}$

**Exercise 4** For each subset  $\Phi \subseteq \wp(\{\alpha, \beta\})$ , try to find a set of rules  $R$  such that  $\Phi$  is the set of all consistent sets for  $R$ .

**Exercise 5** Derive the following additional rules from the basic ones of the Natural Deduction calculus.

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi} \quad \frac{\Gamma \vdash \varphi}{\Gamma, \Delta \vdash \varphi} \quad \frac{\Gamma, \neg\varphi \vdash \neg\psi}{\Gamma \vdash \psi \rightarrow \varphi} \quad \frac{\Gamma, \neg\varphi \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

**Exercise 6** Find a rule for proofs by induction.

$$\frac{\dots \quad \dots}{\forall x \varphi(x)} \quad \text{where } \varphi(x) \text{ is a formula talking about natural numbers.}$$