IA168 Algorithmic Game Theory

Tomáš Brázdil

Organization of This Course

Sources:

- Lectures (slides, notes)
  - based on several sources
  - slides are prepared for lectures, some stuff on greenboard ($\Rightarrow$ attend the lectures)

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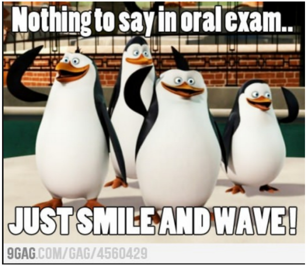
- Books:
  - Nisan/Roughgarden/Tardos/Vazirani, **Algorithmic Game Theory**, Cambridge University, 2007.
    Available online for free:
    http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf
  - Tadelis, **Game Theory: An Introduction**, Princeton University Press, 2013

(I use various resources, so please, attend the lectures)

Evaluation

- **Oral exam**
- Homework



  - 4 times homework
  - A "computer" game

What is Algorithmic Game Theory?

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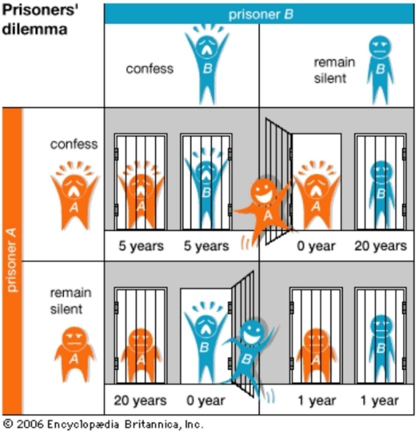
What does the "algorithmic" mean?

- It means that we are "concerned with the computational questions that arise in game theory, and that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games."

Let's have a look at some examples ....

Prisoner’s Dilemma



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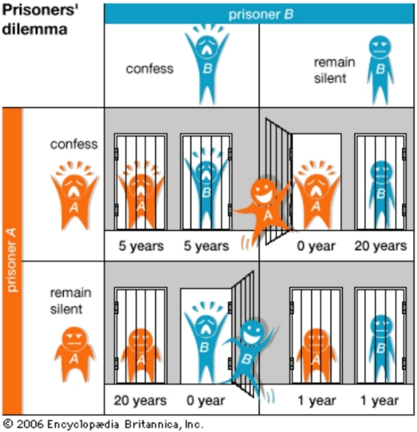

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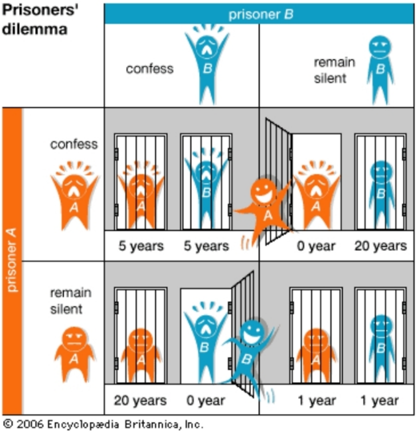

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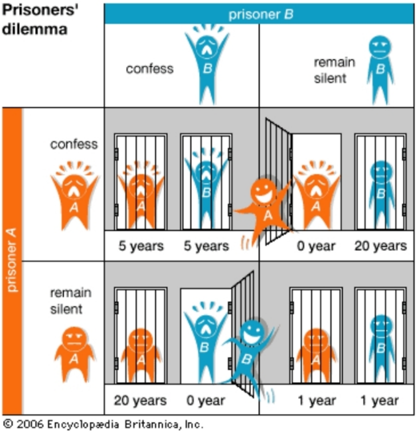

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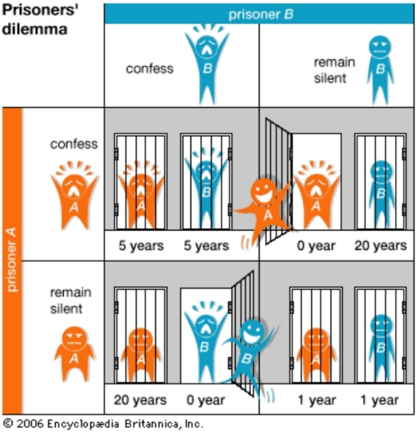

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The problem: What would the suspects do?

Prisoner's Dilemma – Solution(?)

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C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Rational "row" suspect (or his adviser) may reason as follows:

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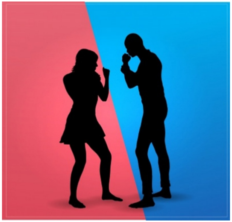
Are there always "dominant" strategies?

Nash equilibria – Battle of Sexes



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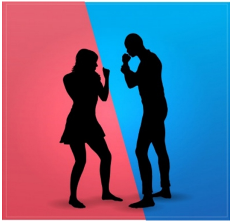

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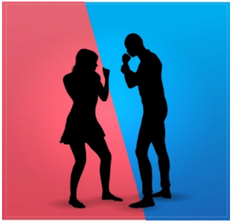

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- The husband would like to go to the football game. The wife would like to go to the opera. Both would prefer to go to the same place rather than different ones.

If they cannot communicate, where should they go?

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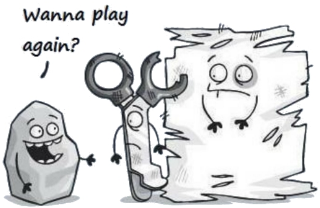
(O,O) is an example of a *Nash equilibrium* (as is (F,F))

Mixed Equilibria – Rock-Paper-Scissors

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$



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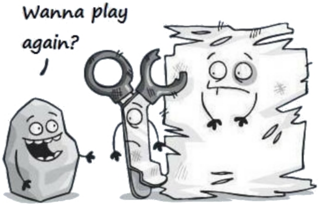

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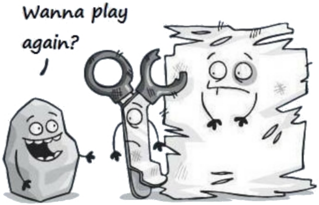

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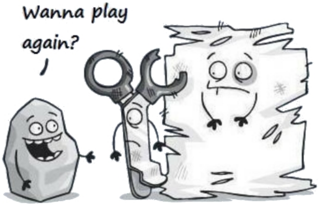

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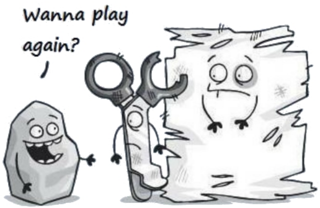

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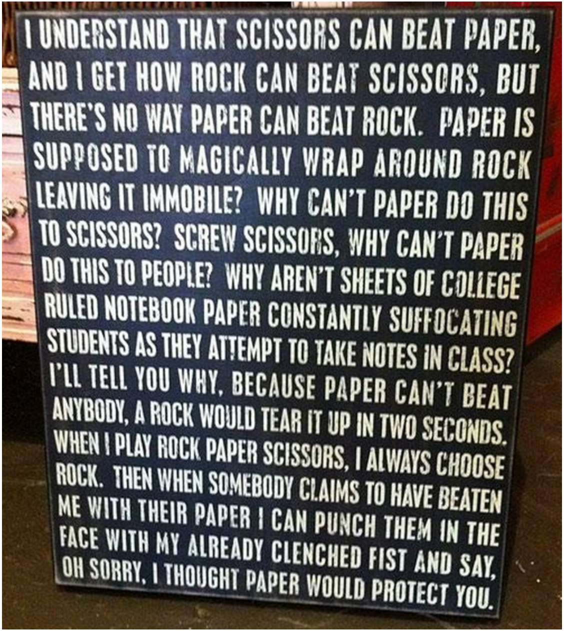
How to algorithmically solve games in mixed strategies? (we shall use probability theory and linear programming)

Philosophical Issues in Games



Dynamic Games

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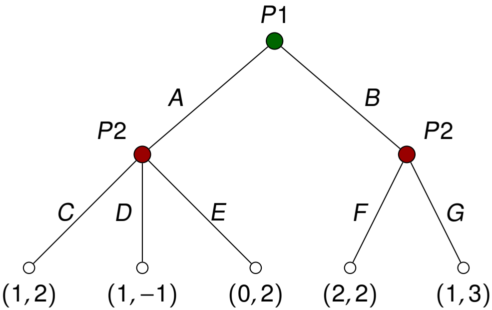

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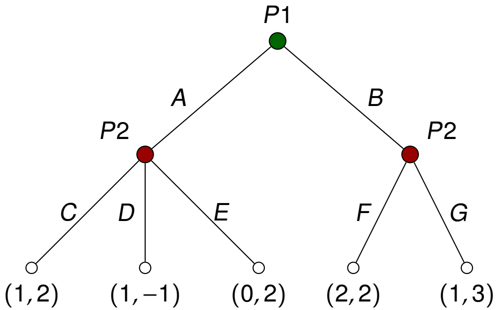

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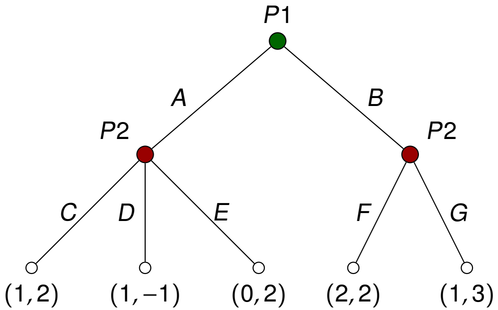

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What is their relationship to the strategic form games?

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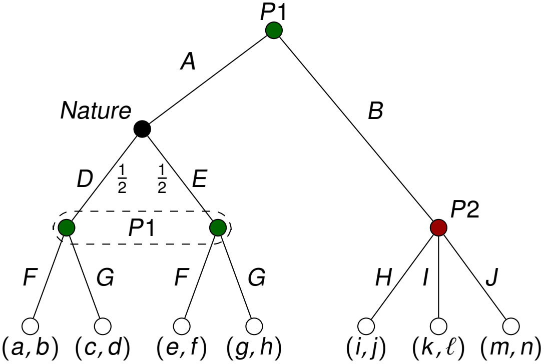

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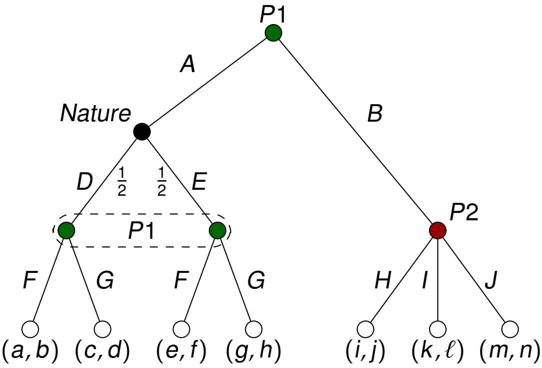

Again, how to solve such games?

Games of Incomplete Information

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In all previous games the players knew all details of the game they played, and this fact was a “common knowledge”. This is not always the case.

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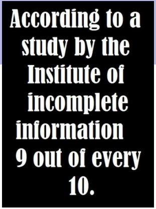

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$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & b_1 > b_2 \\ \frac{1}{2}(v_1 - b_1) & b_1 = b_2 \\ 0 & b_1 < b_2 \end{cases}$$

Here v_1 is the private value that player 1 assigns to the item and so the player 2 **does not know** u_1**.**

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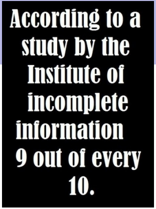

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How to deal with such a game? Assume the “worst” private value? What if we have a partial knowledge about the private values?

Inefficiency of Equilibria

In Prisoner's Dilemma, the selfish behavior of suspects (the Nash equilibrium) results in somewhat worse than ideal situation.

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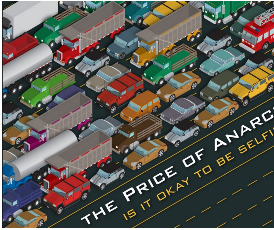
Price of Anarchy is the maximum ratio between values of equilibria and the value of an optimal solution.

Inefficiency of Equilibria – Selfish Routing

Consider a transportation system where many agents are trying to get from some initial location to a destination. Consider the welfare to be the average time for an agent to reach the destination. There are two versions:



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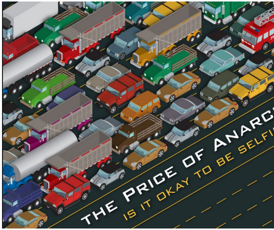

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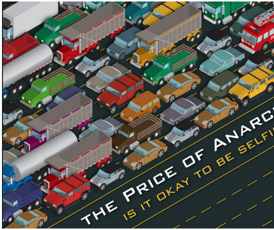

- "Centralized": A central authority tells each agent where to go.
- "Decentralized": Each agent selfishly minimizes his travel time.

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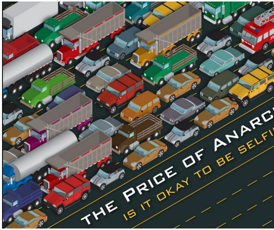

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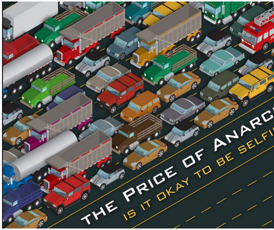
- "Decentralized": Each agent selfishly minimizes his travel time.

Price of Anarchy measure the ratio between average travel time in these two cases.

Inefficiency of Equilibria – Selfish Routing

Consider a transportation system where many agents are trying to get from some initial location to a destination. Consider the welfare to be the average time for an agent to reach the destination. There are two versions:



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Problem: Bound the price of anarchy over all routing games?

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- Games in Logic: modal and temporal logics, Ehrenfeucht-Fraisse games, etc.

Games in Computer Science

Games, the Internet and E-commerce: An extremely active research area at the intersection of CS and Economics

Basic idea: “The internet is a HUGE experiment in interaction between agents (both human and automated)”

How do we set up the rules of this game to harness “socially optimal” results?

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- Remaining time will be devoted to selected topics from extensive form games, games on graphs etc.

Static Games of Complete Information

Strategic-Form Games

Solution concepts

Static Games of Complete Information – Intuition

Proceed in two steps:

1. Each player *simultaneously and independently* chooses a *strategy*. This means that players play without observing strategies chosen by other players.

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A fact E is a *common knowledge* among players $\{1, \ldots, n\}$ if for every sequence $i_1, \ldots, i_k \in \{1, \ldots, n\}$ we have that i_1 knows that i_2 knows that ... i_{k-1} knows that i_k knows E.

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The goal of each player is to maximize his payoff (and this fact is common knowledge).

Strategic-Form Games

To formally represent static games of complete information we define *strategic-form games*.

Definition 2

A game in *strategic-form* (or normal-form) is an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, in which:

- $N = \{1, 2, \ldots, n\}$ is a finite set of *players*.
- S_i is a set of *(pure) strategies* of player i, for every $i \in N$.

  A *strategy profile* is a vector of strategies of all players $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$.

  We denote the set of all strategy profiles by $S = S_1 \times \cdots \times S_n$.
- $u_i : S \to \mathbb{R}$ is a function associating each strategy profile $s = (s_1, \ldots, s_n) \in S$ with the *payoff* $u_i(s)$ to player i, for every player $i \in N$.

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Definition 3

A *zero-sum* game G is one in which for all $s = (s_1, \ldots, s_n) \in S$ we have $u_1(s) + u_2(s) + \cdots + u_n(s) = 0$.

Example: Prisoner's Dilemma

- $N = \{1, 2\}$
- $S_1 = S_2 = \{S, C\}$
- u_1, u_2 are defined as follows:
  - $u_1(C, C) = -5$, $u_1(C, S) = 0$, $u_1(S, C) = -20$, $u_1(S, S) = -1$
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(Is it zero sum?)

We usually write payoffs in the following form:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

or as two matrices:

	C	S
C	-5	0
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	C	S
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Strategic-form game model $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1$
  $u_2(q_1, q_2) = q_2(\kappa - q_1 - q_2) - q_2 c_2$

Solution Concepts

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Example 4

Nash equilibrium is a solution concept. That is, we "solve" games by finding Nash equilibria and declare them to be reasonable outcomes.

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Here 4. implies non-cooperative game theory: Each player is in control of his actions, and he will stick to an action only if he finds it to be in his best interest.

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The basic notion for evaluating "social outcome" is the following

Definition 5

A strategy profile $s \in S$ *Pareto dominates* a strategy profile $s' \in S$ if $u_i(s) \geq u_i(s')$ for all $i \in N$, and $u_i(s) > u_i(s')$ for at least one $i \in N$.
A strategy profile $s \in S$ is *Pareto optimal* if it is not Pareto dominated by any other strategy profile.

We will see more measures of social outcome later.

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We will consider the following solution concepts:

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- iterated elimination of strictly dominated strategies (IESDS)
- rationalizability
- Nash equilibria

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For now, let us concentrate on

pure strategies only!

I.e., no mixed strategies are allowed. We will generalize to mixed setting later.

Notation

- Let $N = \{1, \dots, n\}$ be a finite set and for each $i \in N$ let X_i be a set. Let $X := \prod_{i \in N} X_i = \{(x_1, \dots, x_n) \mid x_j \in X_j, j \in N\}$.
  - For $i \in N$ we define $X_{-i} := \prod_{j \neq i} X_j$, i.e.,

    $$X_{-i} = \{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mid x_j \in X_j, \forall j \neq i\}$$

  - An element of X_{-i} will be denoted by

    $$x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

    We slightly abuse notation and write (x_i, x_{-i}) to denote $(x_1, \dots, x_i, \dots, x_n) \in X$.

Strict Dominance in Pure Strategies

Definition 6

Let $s_i, s_i' \in S_i$ be strategies of player i. Then s_i' is *strictly dominated* by s_i (write $s_i \succ s_i'$) if for any possible combination of the other players' strategies, $s_{-i} \in S_{-i}$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

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Claim 1
An intelligent and rational player will never play a strictly dominated strategy.

Clearly, intelligence implies that the player should recognize dominated strategies, rationality implies that the player will avoid playing them.

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Is the strictly dominant strategy equilibrium always Pareto optimal?

Examples

In the Prisoner's dilemma:

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In the Battle of Sexes:

	O	F
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Indiana Jones and the Last Crusade

(Taken from Dixit & Nalebuff's "The Art of Strategy" and a lecture of Robert Marks)

Indiana Jones, his father, and the Nazis have all converged at the site of the Holy Grail. The two Joneses refuse to help the Nazis reach the last step. So the Nazis shoot Indiana's dad. Only the healing power of the Holy Grail can save the senior Dr. Jones from his mortal wound. Suitably motivated, Indiana leads the way to the Holy Grail. But there is one final challenge. He must choose between literally scores of chalices, only one of which is the cup of Christ. While the right cup brings eternal life, the wrong choice is fatal. The Nazi leader impatiently chooses a beautiful gold chalice, drinks the holy water, and dies from the sudden death that follows from the wrong choice. Indiana picks a wooden chalice, the cup of a carpenter. Exclaiming "There's only one way to find out" he dips the chalice into the font and drinks what he hopes is the cup of life. Upon discovering that he has chosen wisely, Indiana brings the cup to his father and the water heals the mortal wound.

Indiana Jones and the Last Crusade (cont.)

Indy Goofed

- Although this scene adds excitement, it is somewhat embarrassing that such a distinguished professor as Dr. Indiana Jones would overlook his dominant strategy.
- He should have given the water to his father without testing it first.
  - If Indiana has chosen the right cup, his father is still saved.
  - If Indiana has chosen the wrong cup, then his father dies but Indiana is spared.
- Testing the cup before giving it to his father doesn't help, since if Indiana has made the wrong choice, there is no second chance – Indiana dies from the water and his father dies from the wound.

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As rationality is a common knowledge, everyone knows that everyone knows that the game is effectively smaller.

Thus everyone knows, that nobody will play strictly dominated strategies in the smaller game (and such strategies may indeed exist).

Iterated Strict Dominance in Pure Strategies

We know that no rational player ever plays strictly dominated strategies.

As each player knows that each player is rational, each player knows that his opponents will not play strictly dominated strategies and thus all opponents know that *effectively* they are facing a "smaller" game.

As rationality is a common knowledge, everyone knows that everyone knows that the game is effectively smaller.

Thus everyone knows, that nobody will play strictly dominated strategies in the smaller game (and such strategies may indeed exist).

Because it is a common knowledge that all players will perform this kind of reasoning again, the process can continue until no more strictly dominated strategies can be eliminated.

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
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Remark: If all S_i are *finite*, then in 2. we may remove only some of the strictly dominated strategies (not necessarily all). The result is *not* affected by the order of elimination since strictly dominated strategies remain strictly dominated even after removing some other strictly dominated strategies.

IESDS Examples

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all strategies survive all rounds (i.e. IESDS $\equiv$ anything may happen, sorry)

A Bit More Interesting Example

	L	C	R
L	$4,3$	$5,1$	$6,2$
C	$2,1$	$8,4$	$3,6$
R	$3,0$	$9,6$	$2,8$

IESDS on greenboard!

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Hotelling (1929) and Downs (1957)

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- Payoff: The number of voters for the candidate, each candidate (selfishly) strives to maximize this number

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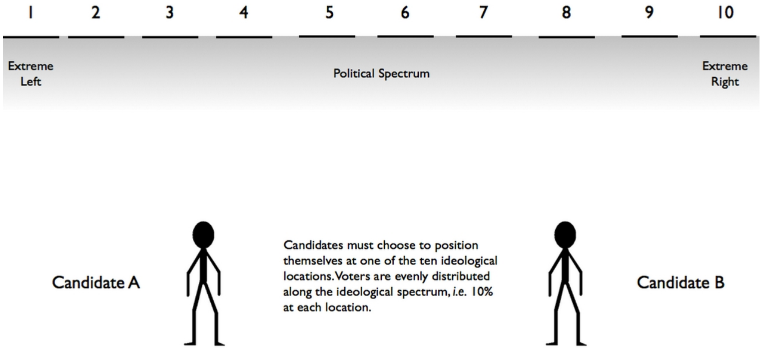

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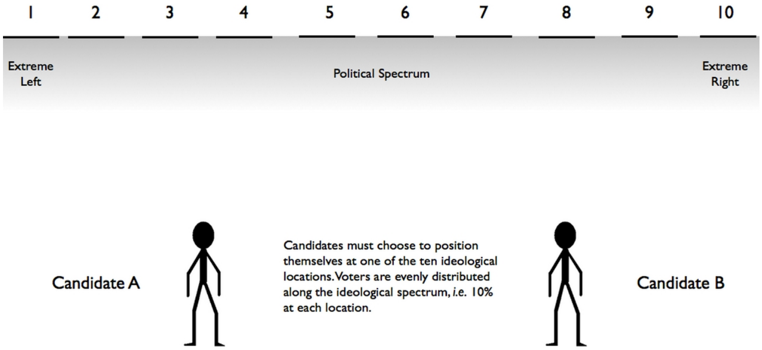

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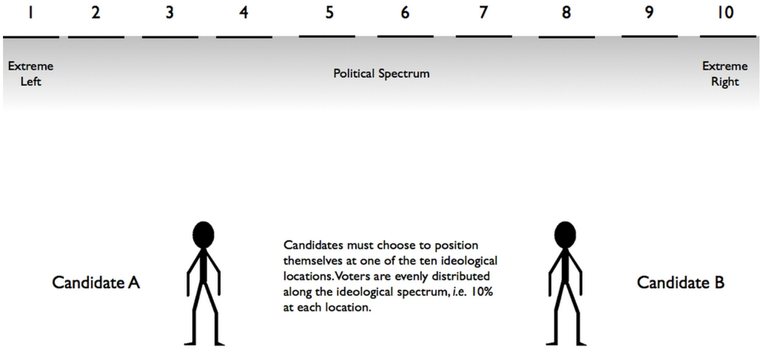

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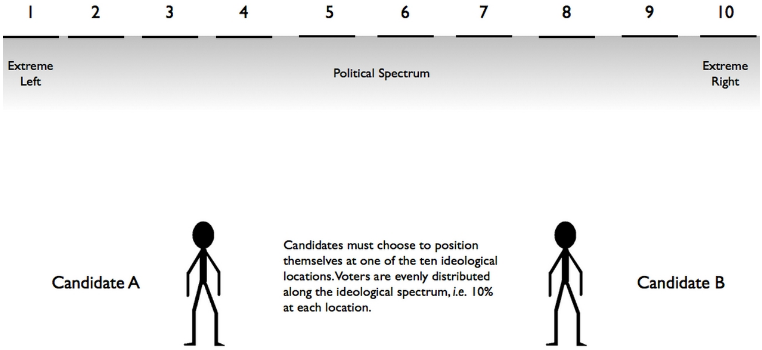

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- $\ldots$
- only $5, 6$ survive IESDS

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Let us formalize this type of reasoning ....

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A strategy $s_i \in S_i$ of player i is a *best response* to a belief $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i$$

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A rational player never plays any strategy that is never best response.

Best Response vs Strict Dominance

Proposition 1

If s_i is strictly dominated for player i, then it is never best response.

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The opposite does not have to be true in pure strategies:

	X	Y
A	$1, 1$	$1, 1$
B	$2, 1$	$0, 1$
C	$0, 1$	$2, 1$

Here A is never best response but is strictly dominated neither by B, nor by C.

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
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(Warning: For some reasons, rationalizable strategies are almost always defined using mixed strategies!)

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all strategies are rationalizable.

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
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Since $q_2 \geq 0$, we obtain that q_1 is never best response iff $q_1 > \theta/2$.
Similarly q_2 is never best response iff $q_2 > \theta/2$.

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Solve $\frac{\delta u_1}{\delta q_1} = \theta - 2q_1 - q_2 = 0$, which gives that $q_1 = (\theta - q_2)/2$ is the only best response of player 1 to q_2.
Similarly, $q_2 = (\theta - q_1)/2$ is the only best response of player 2 to q_1.

Since $q_2 \geq 0$, we obtain that q_1 is never best response iff $q_1 > \theta/2$.
Similarly q_2 is never best response iff $q_2 > \theta/2$.

Thus $R_1^1 = R_2^1 = [0, \theta/2]$.

Cournot Duopoly

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- $N = \{1, 2\}$
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Since $q_2 \in R^1_2 = [0, \theta/2]$, we obtain that q_1 is never best response iff $q_1 \in [0, \theta/4)$
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Hence, $\lim_{k \to \infty} \ell_k = \lim_{k \to \infty} r_k = \theta/3$ and thus $(\theta/3, \theta/3)$ is the only rationalizable equilibrium.

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Are $q_i = \theta/3$ Pareto optimal?

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Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

Are $q_i = \theta/3$ Pareto optimal? NO!

$$u_1(\theta/3, \theta/3) = u_2(\theta/3, \theta/3) = \theta^2/9$$

but

$$u_1(\theta/4, \theta/4) = u_2(\theta/4, \theta/4) = \theta^2/8$$

IESDS vs Rationalizability in Pure Strategies

Theorem 15
Assume that S is finite. Then for all k we have that $R_i^k \subseteq D_i^k$. That is, in particular, all rationalizable strategies survive IESDS.

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Thus s_i is not strictly dominated in G_{DS}^k and $s_i \in D_i^{k+1}$. □

Pinning Down Beliefs – Nash Equilibria

Criticism of previous approaches:

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Here all strategies are equally reasonable according to the above concepts.

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But are all strategy profiles really equally reasonable?

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Note that if player 1 believes that player 2 plays O, then playing O is reasonable, and if player 2 believes that player 1 plays F, then playing F is reasonable. But such **beliefs cannot be correct together**!

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By Claim 3, each player plays a best response to his beliefs.

Is (O, F) as reasonable as (O, O) in this respect?

Note that if player 1 believes that player 2 plays O, then playing O is reasonable, and if player 2 believes that player 1 plays F, then playing F is reasonable. But such **beliefs cannot be correct together**!

(O, O) can be obtained as a profile where each player plays the best response to his belief and the **beliefs are correct**.

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A pure-strategy profile $s^* = (s_1^*, \dots, s_n^*) \in S$ is a (pure) Nash equilibrium if s_i^* is a best response to s_{-i}^* for each $i \in N$, that is

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Note that this definition is equivalent to the previous one in the sense that s_{-i}^* may be considered as the (consistent) belief of player i to which he plays a best response s_i^*

Nash Equilibria Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
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In Cournot Duopoly, $(\theta/3, \theta/3)$ is the only Nash equilibrium.

(Best response relations: $q_1 = (\theta - q_2)/2$ and $q_2 = (\theta - q_1)/2$ are both satisfied only by $q_1 = q_2 = \theta/3$)

Example: Stag Hunt

Story:

- Two (in some versions more than two) hunters, players 1 and 2, can each choose to hunt

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This is supposed to explain that in real world there are societies that have similar endowments, access to technology and physical environment but have very different achievements, all because of self-fulfilling beliefs (or *norms* of behavior).

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So it seems to be rational to expect (H, H) (?)

Nash Equilibria vs Previous Concepts

Theorem 17

1. *If s^* is a strictly dominant strategy equilibrium, then it is the unique Nash equilibrium.*
2. *Each Nash equilibrium is rationalizable and survives IESDS.*
3. *If S is finite, neither rationalizability, nor IESDS creates new Nash equilibria.*

Proof: Homework!

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Proof: Homework!

Corollary 18

Assume that S is finite. If rationalizability or IESDS result in a unique strategy profile, then this profile is a Nash equilibrium.

Interpretations of Nash Equilibria

Except the two definitions, usual interpretations are following:

- When the goal is to give advice to all of the players in a game (i.e., to advise each player what strategy to choose), any advice that was not an equilibrium would have the unsettling property that there would always be some player for whom the advice was bad, in the sense that, if all other players followed the parts of the advice directed to them, it would be better for some player to do differently than he was advised. If the advice is an equilibrium, however, this will not be the case, because the advice to each player is the best response to the advice given to the other players.

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- When the goal is prediction rather than prescription, a Nash equilibrium can also be interpreted as a potential stable point of a dynamic adjustment process in which individuals adjust their behavior to that of the other players in the game, searching for strategy choices that will give them better results.

Static Games of Complete Information

Mixed Strategies

Let’s Mix It

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Example: Rock-Paper-sCissors

	R	P	C
R	$0, 0$	$-1, 1$	$1, -1$
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How to solve this?

Let the players randomize their choice of pure strategies ....

Probability Distributions

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Example 20

Consider $A = \{a, b, c\}$ and a function $\sigma : A \to [0,1]$ such that $\sigma(a) = \frac{1}{4}$, $\sigma(b) = \frac{3}{4}$, and $\sigma(c) = 0$. Then $\sigma \in \Delta(A)$ and $supp(\sigma) = \{a, b\}$.

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For example, in rock-paper-scissors, the pure strategy R corresponds to σ_i which satisfies $\sigma_i(X) = \begin{cases} 1 & X = R \\ 0 & \text{otherwise} \end{cases}$

Mixed Strategy Profiles

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$$\sigma_{-i}(s_{-i}) := \prod_{k \neq i}^{n} \sigma_k(s_k)$$

is the probability that the opponents of player i choose $s_{-i} \in S_{-i}$ when they play according to the mixed strategy profile $\sigma_{-i} \in \Sigma_{-i}$.

(We abuse notation a bit here: σ denotes two things, a vector of mixed strategies as well as a probability distribution on S (the same for σ_{-i})

Mixed Strategies – Example

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Consider a mixed strategy profile (σ_1, σ_2) where $\sigma_1 = (\frac{1}{2}(R), \frac{1}{3}(P), \frac{1}{6}(C))$ and $\sigma_2 = (\frac{1}{3}(R), \frac{2}{3}(P), 0(C))$.

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$\sigma_1 = (\frac{1}{2}(R), \frac{1}{3}(P), \frac{1}{6}(C))$ and $\sigma_2 = (\frac{1}{3}(R), \frac{2}{3}(P), 0(C))$.
Then the probability $\sigma(R, P)$ that the pure strategy profile (R, P) will be chosen by players playing the mixed profile (σ_1, σ_2) is

$$\sigma_1(R) \cdot \sigma_2(P) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

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Definition 22
The *expected payoff* of player i under a mixed strategy profile $\sigma \in \Sigma$ is

$$u_i(\sigma) := \sum_{s \in S} \sigma(s) u_i(s) \qquad \left(= \sum_{s \in S} \prod_{k=1}^{n} \sigma_k(s_k) u_i(s) \right)$$

I.e., it is the "weighted average" of what player i wins under each pure strategy profile s, weighted by the probability of that profile.

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Assumption: Every rational player strives to maximize his own expected payoff.
(This assumption is not always completely convincing ...)

Expected Payoff – Example

Matching Pennies:

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Each player secretly turns a penny to heads or tails, and then they reveal their choices simultaneously. If the pennies match, player 1 (row) wins, if they do not match, player 2 (column) wins.

Consider $\sigma_1 = (\frac{1}{3}(H), \frac{2}{3}(T))$ and $\sigma_2 = (\frac{1}{4}(H), \frac{3}{4}(T))$

$$\begin{aligned} u_1(\sigma_1, \sigma_2) &= \sum_{(X,Y)\in\{H,T\}^2} \sigma_1(X)\sigma_2(Y)u_1(X,Y) \\ &= \frac{1}{3}\frac{1}{4}1 + \frac{1}{3}\frac{3}{4}(-1) + \frac{2}{3}\frac{1}{4}(-1) + \frac{2}{3}\frac{3}{4}1 = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} u_2(\sigma_1, \sigma_2) &= \sum_{(X,Y)\in\{H,T\}^2} \sigma_1(X)\sigma_2(Y)u_2(X,Y) \\ &= \frac{1}{3}\frac{1}{4}(-1) + \frac{1}{3}\frac{3}{4}1 + \frac{2}{3}\frac{1}{4}1 + \frac{2}{3}\frac{3}{4}(-1) = -\frac{1}{6} \end{aligned}$$

"Decomposition" of Expected Payoff

Consider the matching pennies example from the previous slide:

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

together with some mixed strategies σ_1 and σ_2.

We prove the following important property of the expected payoff:

$$u_1(\sigma_1,\sigma_2) = \sum_{X\in\{H,T\}} \sigma_1(X)u_1(X,\sigma_2)$$

An intuition behind this equality is following:

- $u_1(\sigma_1,\sigma_2)$ is the expected payoff of player 1 in the following experiment: Both players simultaneously and independently choose their pure strategies X, Y according to σ_1, σ_2, resp., and then player 1 collects his payoff $u_1(X, Y)$.
- $\sum_{X\in\{H,T\}} \sigma_1(X)u_1(X,\sigma_2)$ is the expected payoff of player 1 in the following: Player 1 chooses his *pure* strategy X and then uses it against the mixed strategy σ_2 of player 2. Then player 2 chooses Y according to σ_2 *independently of* X, and player 1 collects the payoff $u_1(X, Y)$.

As Y does not depend on X in neither experiment, we obtain the above equality of expected payoffs.

"Decomposition" of Expected Payoff

Consider the matching pennies example from the previous slide:

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

together with some mixed strategies σ_1 and σ_2.

A formal proof is straightforward:

$$
\begin{aligned}
u_1(\sigma_1, \sigma_2) &= \sum_{(X,Y)\in\{H,T\}^2} \sigma_1(X)\sigma_2(Y)u_1(X,Y) \\
&= \sum_{X\in\{H,T\}} \sum_{Y\in\{H,T\}} \sigma_1(X)\sigma_2(Y)u_1(X,Y) \\
&= \sum_{X\in\{H,T\}} \sigma_1(X) \sum_{Y\in\{H,T\}} \sigma_2(Y)u_1(X,Y) \\
&= \sum_{X\in\{H,T\}} \sigma_1(X)u_1(X, \sigma_2)
\end{aligned}
$$

(In the last equality we used the fact that X is identified with a mixed strategy assigning one to X.)

"Decomposition" of Expected Payoff

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	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

together with some mixed strategies σ_1 and σ_2.

Similarly,

$$\begin{aligned}
u_1(\sigma_1, \sigma_2) &= \sum_{(X,Y)\in\{H,T\}^2} \sigma_1(X)\sigma_2(Y)u_1(X,Y) \\
&= \sum_{X\in\{H,T\}} \sum_{Y\in\{H,T\}} \sigma_1(X)\sigma_2(Y)u_1(X,Y) \\
&= \sum_{Y\in\{H,T\}} \sum_{X\in\{H,T\}} \sigma_1(X)\sigma_2(Y)u_1(X,Y) \\
&= \sum_{Y\in\{H,T\}} \sigma_2(Y) \sum_{X\in\{H,T\}} \sigma_1(X)u_1(X,Y) \\
&= \sum_{Y\in\{H,T\}} \sigma_2(Y)u_1(\sigma_1, Y)
\end{aligned}$$

Expected Payoff – "Decomposition" in General

Lemma 23

For every mixed strategy profile $\sigma \in \Sigma$ *and every* $k \in N$ *we have*

$$u_i(\sigma) = \sum_{s_k \in S_k} \sigma_k(s_k) \cdot u_i(s_k, \sigma_{-k}) = \sum_{s_{-k} \in S_{-k}} \sigma_{-k}(s_{-k}) \cdot u_i(\sigma_k, s_{-k})$$

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Proof:

$$\begin{aligned}
u_i(\sigma) = \sum_{s \in S} \sigma(s) u_i(s) &= \sum_{s \in S} \prod_{\ell=1}^{n} \sigma_\ell(s_\ell) u_i(s) \\
&= \sum_{s \in S} \sigma_k(s_k) \prod_{\ell \neq k}^{n} \sigma_\ell(s_\ell) u_i(s) \\
&= \sum_{s_k \in S_k} \sum_{s_{-k} \in S_{-k}} \sigma_k(s_k) \prod_{\ell \neq k}^{n} \sigma_\ell(s_\ell) u_i(s_k, s_{-k}) \\
&= \sum_{s_k \in S_k} \sum_{s_{-k} \in S_{-k}} \sigma_k(s_k) \sigma_{-k}(s_{-k}) u_i(s_k, s_{-k})
\end{aligned}$$

Proof of Lemma 23 (cont.)

The first equality:

$$
\begin{aligned}
u_i(\sigma) &= \sum_{s_k \in S_k} \sum_{s_{-k} \in S_{-k}} \sigma_k(s_k)\sigma_{-k}(s_{-k})u_i(s_k, s_{-k}) \\
&= \sum_{s_k \in S_k} \sigma_k(s_k) \sum_{s_{-k} \in S_{-k}} \sigma_{-k}(s_{-k})u_i(s_k, s_{-k}) \\
&= \sum_{s_k \in S_k} \sigma_k(s_k) u_i(s_k, \sigma_{-k})
\end{aligned}
$$

Proof of Lemma 23 (cont.)

The first equality:

$$u_i(\sigma) = \sum_{s_k \in S_k} \sum_{s_{-k} \in S_{-k}} \sigma_k(s_k)\sigma_{-k}(s_{-k})u_i(s_k, s_{-k})$$

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$$= \sum_{s_k \in S_k} \sigma_k(s_k)u_i(s_k, \sigma_{-k})$$

The second equality:

$$u_i(\sigma) = \sum_{s_k \in S_k} \sum_{s_{-k} \in S_{-k}} \sigma_k(s_k)\sigma_{-k}(s_{-k})u_i(s_k, s_{-k})$$

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Expected Payoff – Pure Strategy Bounds

Corollary 24

For all $i, k \in N$ and $\sigma \in \Sigma$ we have that

- $\min_{s_k \in S_k} u_i(s_k, \sigma_{-k}) \leq u_i(\sigma) \leq \max_{s_k \in S_k} u_i(s_k, \sigma_{-k})$
- $\min_{s_{-k} \in S_{-k}} u_i(\sigma_k, s_{-k}) \leq u_i(\sigma) \leq \max_{s_{-k} \in S_{-k}} u_i(\sigma_k, s_{-k})$

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Proof.

We prove $u_i(\sigma) \leq \max_{s_k \in S_k} u_i(s_k, \sigma_{-k})$ the rest is similar. Define $B := \max_{s_k \in S_k} u_i(s_k, \sigma_{-k})$. Then

$$
\begin{aligned}
u_i(\sigma) &= \sum_{s_k \in S_k} \sigma_k(s_k) \cdot u_i(s_k, \sigma_{-k}) \\
&\leq \sum_{s_k \in S_k} \sigma_k(s_k) \cdot B \\
&= B
\end{aligned}
$$

□

Solution Concepts

We revisit the following solution concepts in mixed strategies:

- strict dominant strategy equilibrium
- IESDS equilibrium
- rationalizable equilibria
- Nash equilibria

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Solution Concepts

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- strict dominant strategy equilibrium
- IESDS equilibrium
- rationalizable equilibria
- Nash equilibria

From now on, when I say a *strategy* I implicitly mean a

mixed strategy.

In order to deal with efficiency issues we assume that the size of the game G is defined by $|G| := |N| + \sum_{i \in N} |S_i| + \sum_{i \in N} |u_i|$ where $|u_i| = \sum_{s \in S} |u_i(s)|$ and $|u_i(s)|$ is the length of a binary encoding of $u_i(s)$ (we assume that rational numbers are encoded as quotients of two binary integers)

Note that, in particular, $|G| > |S|$.

Strict Dominance in Mixed Strategies

Definition 25

Let $\sigma_i, \sigma'_i \in \Sigma_i$ be (mixed) strategies of player i. Then σ'_i is *strictly dominated* by σ_i (write $\sigma'_i \prec \sigma_i$) if

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma'_i, \sigma_{-i}) \quad \text{for all } \sigma_{-i} \in \Sigma_{-i}$$

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Example 26

	X	Y
A	3	0
B	0	3
C	1	1

Is there a strictly dominated strategy?

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Example 26

	X	Y
A	3	0
B	0	3
C	1	1

Is there a strictly dominated strategy?

Question: Is there a game with at least one strictly dominated strategy but without strictly dominated *pure* strategies?

Strictly Dominant Strategy Equilibrium

Definition 27

$\sigma_i \in \Sigma_i$ is *strictly dominant* if every other mixed strategy of player *i* is strictly dominated by σ_i.

Strictly Dominant Strategy Equilibrium

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A strategy profile $\sigma \in \Sigma$ is a *strictly dominant strategy equilibrium* if $\sigma_i \in \Sigma_i$ is strictly dominant for all $i \in N$.

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Proposition 2

If the strictly dominant strategy equilibrium exists, it is unique, all its strategies are pure, and rational players will play it.

Proof.

Let $\sigma^* = (\sigma^*_1, \ldots, \sigma^*_n) \in \Sigma_i$ be a strictly dominant strategy equilibrium.

By Corollary 24, for every $i \in N$, there must exist $s_i \in S_i$ such that $u_i(\sigma^*) \leq u_i(s_i, \sigma^*_{-i})$.

But then $\sigma^*_i = s_i$ since σ^*_i is strictly dominant.

□

Computing Strictly Dominant Strategy Equilibrium

How to decide whether there is a strictly dominant strategy equilibrium $s = (s_1, \ldots, s_n) \in S$?

I.e. whether for a given $s_i \in S_i$, all $\sigma_i \in \Sigma_i \setminus \{s_i\}$ and all $\sigma_{-i} \in \Sigma_{-i}$:

$$u_i(s_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$$

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$$u_i(s_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$$

There are some serious issues here:

Obviously there are uncountably many possible σ_i and σ_{-i}.

$u_i(\sigma_i, \sigma_{-i})$ is nonlinear, and for more than two players even $u_i(s_i, \sigma_{-i})$ is nonlinear in probabilities assigned to pure strategies.

Computing Strictly Dominant Strategy Equilibrium

First, we prove the following useful proposition using Lemma 23:

Lemma 29

σ'_i *strictly dominates* σ_i **iff** *for all* pure strategy profiles $s_{-i} \in S_{-i}$:

$$u_i(\sigma'_i, s_{-i}) \quad > \quad u_i(\sigma_i, s_{-i}) \tag{1}$$

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$$u_i(\sigma'_i, s_{-i}) \quad > \quad u_i(\sigma_i, s_{-i}) \tag{1}$$

Proof.

'⇒' direction is trivial, let us prove '⇐'. Assume that (1) is true for all pure strategy profiles $s_{-i} \in S_{-i}$. Then, by Lemma 23,

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(\sigma_i, s_{-i}) < \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(\sigma'_i, s_{-i}) = u_i(\sigma'_i, \sigma_{-i})$$

holds for all mixed strategy profiles $\sigma_{-i} \in \Sigma_{-i}$. □

In other words, it suffices to check the strict dominance only with respect to all *pure* profiles of opponents.

Computing Strictly Dominant Strategy Equilibrium

How to decide whether for a given $s_i \in S_i$, all $\sigma_i \in \Sigma_i \smallsetminus \{s_i\}$ and all $s_{-i} \in S_{-i}$ we have $u_i(s_i, s_{-i}) > u_i(\sigma_i, s_{-i})$?

Lemma 30

$u_i(s_i, s_{-i}) > u_i(\sigma_i, s_{-i})$ for all $\sigma_i \in \Sigma_i \smallsetminus \{s_i\}$ and all $s_{-i} \in S_{-i}$ **iff** *$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all $s_i' \in S_i \smallsetminus \{s_i\}$ and all $s_{-i} \in S_{-i}$.*

Proof.

'$\Rightarrow$' direction is trivial, let us prove '$\Leftarrow$'. Assume $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all $s_i' \in S_i \smallsetminus \{s_i\}$ and all $s_{-i} \in S_{-i}$. Given $\sigma_i \in \Sigma_i \smallsetminus \{s_i\}$, we have by Lemma 23,

$$u_i(\sigma_i, s_{-i}) = \sum_{s_i' \in S_i} \sigma_i(s_i') u_i(s_i', s_{-i}) < \sum_{s_i' \in S_i} \sigma_i(s_i') u_i(s_i, s_{-i}) = u_i(s_i, s_{-i})$$

The inequality follows from our assumption and the fact that $\sigma_i(s_i') > 0$ for at least one $s_i' \neq s_i$ (due to $\sigma_i \in \Sigma_i \smallsetminus \{s_i\}$). □

Computing Strictly Dominant Strategy Equilibrium

How to decide whether for a given $s_i \in S_i$, all $\sigma_i \in \Sigma_i \setminus \{s_i\}$ and all $s_{-i} \in S_{-i}$ we have $u_i(s_i, s_{-i}) > u_i(\sigma_i, s_{-i})$?

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$u_i(s_i, s_{-i}) > u_i(\sigma_i, s_{-i})$ for all $\sigma_i \in \Sigma_i \setminus \{s_i\}$ and all $s_{-i} \in S_{-i}$ **iff** *$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all $s_i' \in S_i \setminus \{s_i\}$ and all $s_{-i} \in S_{-i}$.*

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The inequality follows from our assumption and the fact that $\sigma_i(s_i') > 0$ for at least one $s_i' \neq s_i$ (due to $\sigma_i \in \Sigma_i \setminus \{s_i\}$). □

Thus it suffices to check whether $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and all $s_{-i} \in S_{-i}$. This can easily be done in time polynomial w.r.t. $|G|$.

IESDS in Mixed Strategies

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting the pure strategy sets to D_i^k, $i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are *not* strictly dominated in G_{DS}^k by *mixed strategies*.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ *survives IESDS* if $s_i \in D_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 31
A strategy profile $s = (s_1, \ldots, s_n) \in S$ is an *IESDS equilibrium* if each s_i survives IESDS.

IESDS – Algorithm

Note that in step 2 it is not sufficient to consider pure strategies. Consider the following zero sum game:

	X	Y
A	3	0
B	0	3
C	1	1

IESDS – Algorithm

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	X	Y
A	3	0
B	0	3
C	1	1

C is strictly dominated by $(\sigma_1(A), \sigma_1(B), \sigma_1(C)) = (\frac{1}{2}, \frac{1}{2}, 0)$ but no strategy is strictly dominated in pure strategies.

IESDS – Algorithm

However, there are uncountably many mixed strategies that may dominate a given pure strategy ...

IESDS – Algorithm

However, there are uncountably many mixed strategies that may dominate a given pure strategy ...

But $u_i(\sigma) = u_i(\sigma_1, \ldots, \sigma_n)$ is linear in each σ_k (if σ_{-k} is kept fixed)!

Indeed, assuming w.l.o.g. that $S_k = \{1, \ldots, m_k\}$,

$$u_i(\sigma) = \sum_{s_k \in S_k} \sigma_k(s_k) \cdot u_i(s_k, \sigma_{-k}) = \sum_{\ell=1}^{m_k} \sigma_k(\ell) \cdot u_i(\ell, \sigma_{-k})$$

is the scalar product of the vector $\sigma_k = (\sigma_k(1), \ldots, \sigma_k(m_k))$ with the vector $(u_i(1, \sigma_{-k}), \ldots, u_i(m_k, \sigma_{-k}))$, which is linear.

So to decide strict dominance, we use linear programming ...

Intermezzo: Linear Programming

Linear programming is a technique for optimization of a linear objective function, subject to linear (non-strict) inequality constraints.

Formally, a linear program in so called *canonical form* looks like this:

$$\text{maximize} \sum_{j=1}^{m} c_j x_j \qquad \textit{(objective function)}$$

$$\text{subject to} \sum_{j=1}^{m} a_{ij} x_j \leq b_i \qquad 1 \leq i \leq n$$

$$x_j \geq 0 \qquad 1 \leq j \leq m \qquad \textit{(constraints)}$$

Here a_{ij}, b_k and c_j are real numbers and x_j's are real variables.

A *feasible solution* is an assignment of real numbers to the variables $x_j, 1 \leq j \leq m$, so that the *constraints* are satisfied.

An *optimal solution* is a feasible solution which maximizes the *objective function* $\sum_{j=1}^{m} c_j x_j$.

Intermezzo: Complexity of Linear Programming

We assume that coefficients a_{ij}, b_k and c_j are encoded in binary (more precisely, as fractions of two integers encoded in binary).

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Theorem 32 (Khachiyan, Doklady Akademii Nauk SSSR, 1979)

There is an algorithm which for any linear program computes an optimal solution in polynomial time.

The algorithm uses so called ellipsoid method.

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For more info see

http://en.wikipedia.org/wiki/Linear_programming#Solvers_and_scripting_.28programming.29_languages

IESDS Algorithm – Strict Dominance Step

So how do we use linear programming to decide strict dominance in step 2 of IESDS procedure?
I.e. whether for a given s_i there exists σ_i such that for all σ_{-i} we have

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Recall that by Lemma 29 we have that σ_i strictly dominates s_i **iff** for all *pure strategy profiles* $s_{-i} \in S_{-i}$:

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IESDS Algorithm – Strict Dominance Step

Recall that $u_i(\sigma_i, s_{-i}) = \sum_{s'_i \in S_i} \sigma_i(s'_i) u_i(s'_i, s_{-i})$.

So to decide whether $s_i \in S_i$ is strictly dominated by some mixed strategy σ_i, it suffices to solve the following system:

$$\sum_{s'_i \in S_i} x_{s'_i} \cdot u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \qquad s_{-i} \in S_{-i}$$

$$x_{s'_i} \geq 0 \qquad s'_i \in S_i$$

$$\sum_{s'_i \in S_i} x_{s'_i} = 1$$

(Here each variable $x_{s'_i}$ corresponds to the probability $\sigma_i(s'_i)$ assigned by the strictly dominant strategy σ_i to s'_i)

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(Here each variable $x_{s'_i}$ corresponds to the probability $\sigma_i(s'_i)$ assigned by the strictly dominant strategy σ_i to s'_i)

Unfortunately, this is a "strict linear program" ... How to deal with the strict inequality?

IESDS Algorithm – Complexity

Introduce a new variable y to be **maximized** under the following constraints:

$$\sum_{s_i' \in S_i} x_{s_i'} \cdot u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) + y \qquad s_{-i} \in S_{-i}$$

$$x_{s_i'} \geq 0 \qquad s_i' \in S_i$$

$$\sum_{s_i' \in S_i} x_{s_i'} = 1$$

$$y \geq 0$$

Now s_i is strictly dominated **iff** a solution maximizing y satisfies $y > 0$

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Now s_i is strictly dominated **iff** a solution maximizing y satisfies $y > 0$

The size of the above program is polynomial in $|G|$.

So the step 2 of IESDS can be executed in polynomial time.

As every iteration of IESDS removes at least one pure strategy, IESDS runs in time polynomial in $|G|$.

IESDS in Mixed Strategie – Example

	X	Y
A	3	0
B	0	3
C	1	1

Let us have a look at the first iteration of IESDS.

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Observe that A, B are not strictly dominated by any mixed strategy.

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Let us have a look at the first iteration of IESDS.

Observe that A, B are not strictly dominated by any mixed strategy.

Let us construct the linear program for deciding whether C is strictly dominated: The program maximizes y under the following constraints:

$$
\begin{aligned}
3x_A + 0x_B + x_C &\geq 1 + y && \text{Row's payoff against } X \\
0x_A + 3x_B + x_C &\geq 1 + y && \text{Row's payoff against } Y \\
x_A, x_B, x_C &\geq 0 \\
x_A + x_B + x_C &= 1 && \text{x's must make a distribution} \\
y &\geq 0
\end{aligned}
$$

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x_A, x_B, x_C &\geq 0 && \\
x_A + x_B + x_C &= 1 && \text{x's must make a distribution} \\
y &\geq 0 &&
\end{aligned}
$$

The maximum $y = \frac{1}{2}$ is attained at $x_A = \frac{1}{2}$ and $x_B = \frac{1}{2}$.

Best Response

Definition 33
A strategy $\sigma_i \in \Sigma_i$ of player i is a *best response* to a strategy profile $\sigma_{-i} \in \Sigma_{-i}$ of his opponents if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \qquad \text{for all } \sigma_i' \in \Sigma_i$$

We denote by $BR_i(\sigma_{-i}) \subseteq \Sigma_i$ the set of all best responses of player i to the strategy profile of opponents $\sigma_{-i} \in \Sigma_{-i}$.

Best Response – Example

Consider a game with the following payoffs of player 1:

	X	Y
A	2	0
B	0	2
C	1	1

- Player 1 (row) plays $\sigma_1 = (a(A), b(B), c(C))$.
- Player 2 (column) plays $(q(X), (1-q)(Y))$ (we write just q).

Compute $BR_1(q)$.

Rationalizability in Mixed Strategies (Two Players)

For simplicity, we temporarily switch to **two-player** setting $N = \{1, 2\}$.

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Definition 34
A *(mixed) belief* of player $i \in \{1, 2\}$ is a mixed strategy σ_{-i} of his opponent.

(A general definition works with so called *correlated beliefs* that are arbitrary distributions on S_{-i}, the notion of the expected payoff needs to be adjusted, we are not going in this direction ....)

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Assumption: *Any rational player with a belief σ_{-i} always plays a best response to σ_{-i}.*

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Assumption: *Any rational player with a belief σ_{-i} always plays a best response to σ_{-i}.*

Definition 35
A strategy $\sigma_i \in \Sigma_i$ of player $i \in \{1, 2\}$ is *never best response* if it is not a best response to any belief σ_{-i}.

No rational player plays a strategy that is never best response.

Rationalizability in Mixed Strategies (Two Players)

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{Rat}^k the game obtained from G by restricting the pure strategy sets to R_i^k, $i \in N$.)

1. Initialize $k = 0$ and $R_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let R_i^{k+1} be the set of all strategies of R_i^k that are *best responses to some (mixed) beliefs in* G_{Rat}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ is *rationalizable* if $s_i \in R_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 36
A strategy profile $s = (s_1, \ldots, s_n) \in S$ is a *rationalizable equilibrium* if each s_i is rationalizable.

Rationalizability vs IESDS (Two Players)

	X	Y
A	3	0
B	0	3
C	1	1

- Player 1 (row) plays $\sigma_1 = (a(A), b(B), c(C))$
- player 2 (column) plays $(q(X), (1 - q)(Y))$ (we write just q)

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What strategies of player 1 are never best responses?

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What strategies of player 1 are strictly dominated?

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Observation: The set of strictly dominated strategies coincides with the set of never best responses!

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What strategies of player 1 are never best responses?

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... and this holds in general for two player games:

Theorem 37

Assume $N = \{1, 2\}$. A pure strategy s_i is never best response to any belief $\sigma_{-i} \in \Sigma_{-i}$ **iff** *s_i is strictly dominated by a strategy $\sigma_i \in \Sigma_i$.*

It follows that a strategy of S_i survives IESDS **iff** it is rationalizable.

(The theorem is true also for an arbitrary number of players but correlated beliefs need to be used.)

Mixed Nash Equilibrium

Definition 38
A mixed-strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*) \in \Sigma$ is a (mixed) Nash equilibrium if σ_i^* is a best response to σ_{-i}^* for each $i \in N$, that is

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \quad \text{for all } \sigma_i \in \Sigma_i \text{ and all } i \in N$$

An interpretation: each σ_{-i}^* can be seen as a belief of player i against which he plays a best response σ_i^*.

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An interpretation: each σ_{-i}^* can be seen as a belief of player i against which he plays a best response σ_i^*.

Given a mixed strategy profile of opponents $\sigma_{-i} \in \Sigma_{-i}$, we denote by $BR_i(\sigma_{-i})$ the set of all $\sigma_i \in \Sigma_i$ that are best responses to σ_{-i}.

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Then σ^* is a Nash equilibrium iff $\sigma_i^* \in BR_i(\sigma_{-i}^*)$ for all $i \in N$.

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Theorem 39 (Nash 1950)

Every finite game in strategic form has a Nash equilibrium.

This is THE fundamental theorem of game theory.

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Player 1 (row) plays $(p(H), (1-p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1-q)(T))$ (we write q).

Compute all Nash equilibria.

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What are the expected payoffs of playing pure strategies for player 1?

$$u_1(H, q) = 2q - 1 \text{ and } u_1(T, q) = 1 - 2q$$

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$u_1(p, q) = pu_1(H, q) + (1-p)u_1(T, q) = p(2q-1) + (1-p)(1-2q)$.

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Then
$u_1(p, q) = pu_1(H, q) + (1-p)u_1(T, q) = p(2q-1) + (1-p)(1-2q).$

We obtain the best-response correspondence BR_1:

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{2} \\ p \in [0, 1] & \text{if } q = \frac{1}{2} \\ p = 1 & \text{if } q > \frac{1}{2} \end{cases}$$

Example: Matching Pennies

	H	T
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T	$-1, 1$	$1, -1$

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Similarly for player 2 :

$$u_2(p, H) = 1 - 2p \text{ and } u_2(p, T) = 2p - 1$$

Example: Matching Pennies

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$$u_2(p, q) = qu_2(p, H) + (1-q)u_2(p, T) = q(1-2p) + (1-q)(2p-1)$$

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Player 1 (row) plays $(p(H), (1-p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1-q)(T))$ (we write q).

Compute all Nash equilibria.

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$u_2(p, q) = qu_2(p, H) + (1-q)u_2(p, T) = q(1-2p) + (1-q)(2p-1)$
We obtain best-response relation BR_2:

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < \frac{1}{2} \\ q \in [0, 1] & \text{if } p = \frac{1}{2} \\ q = 0 & \text{if } p > \frac{1}{2} \end{cases}$$

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
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The only "intersection" of BR_1 and BR_2 is the only Nash equilibrium $\sigma_1 = \sigma_2 = (\frac{1}{2}, \frac{1}{2})$.

Static Games of Complete Information
Mixed Strategies
Computing Nash Equilibria – Support Enumeration

Computing Mixed Nash Equilibria

Lemma 40

$\sigma^* = (\sigma_1^*, \dots, \sigma_n^*) \in \Sigma$ is a Nash equilibrium **iff** *there exist $w_1, \dots, w_n \in \mathbb{R}$ such that the following holds:*

- *For all $i \in N$ and all $s_i \in supp(\sigma_i^*)$ we have $u_i(s_i, \sigma_{-i}^*) = w_i$.*
- *For all $i \in N$ and all $s_i \notin supp(\sigma_i^*)$ we have $u_i(s_i, \sigma_{-i}^*) \leq w_i$.*

Here, the right hand side implies $u_i(\sigma^*) = w_i$.

Proof.

The fact that the right hand side implies $u_i(\sigma^*) = w_i$ follows immediately from Lemma 23:

$$u_i(\sigma^*) = \sum_{s_i \in S_i} \sigma^*(s_i) u_i(s_i, \sigma_{-i}^*) = \sum_{s_i \in supp(\sigma_i^*)} \sigma^*(s_i) u_i(s_i, \sigma_{-i}^*)$$

$$= \sum_{s_i \in supp(\sigma_i^*)} \sigma^*(s_i) w_i = w_i \sum_{s_i \in supp(\sigma_i^*)} \sigma^*(s_i) = w_i$$

Computing Mixed Nash Equilibria

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$\sigma^* = (\sigma^*_1, \ldots, \sigma^*_n) \in \Sigma$ is a Nash equilibrium **iff** *there exist $w_1, \ldots, w_n \in \mathbb{R}$ such that the following holds:*

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Here, the right hand side implies $u_i(\sigma^*) = w_i$.

Proof. (Cont.)

"$\Leftarrow$": Use the first equality of Lemma 23 to obtain for every $i \in N$ and every $\sigma'_i \in \Sigma_i$

$$u_i(\sigma'_i, \sigma^*_{-i}) = \sum_{s_i \in S_i} \sigma'_i(s_i) u_i(s_i, \sigma^*_{-i}) \leq$$

$$\leq \sum_{s_i \in S_i} \sigma'_i(s_i) w_i = \sum_{s_i \in S_i} \sigma'_i(s_i) u_i(\sigma^*) = u_i(\sigma^*)$$

Thus σ^* is a Nash equilibrium.

Computing Mixed Nash Equilibria

Lemma 42

$\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*) \in \Sigma$ is a Nash equilibrium **iff** *there exist $w_1, \ldots, w_n \in \mathbb{R}$ such that the following holds:*

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Proof (Cont.)

Idea for "⇒": Let $w_i := u_i(\sigma^*)$.

Computing Mixed Nash Equilibria

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Here, the right hand side implies $u_i(\sigma^*) = w_i$.

Proof (Cont.)

Idea for "⇒": Let $w_i := u_i(\sigma^*)$.

Clearly, every $i \in N$ and $s_i \in S_i$ satisfy $u_i(s_i, \sigma_{-i}^*) \leq u_i(\sigma^*) = w_i$.

By Corollary 24, there is *at least one $s_i \in supp(\sigma_i^*)$* satisfying $u_i(s_i, \sigma_{-i}^*) = u_i(\sigma^*) = w_i$.

Computing Mixed Nash Equilibria

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Here, the right hand side implies $u_i(\sigma^*) = w_i$.

Proof (Cont.)

Idea for "$\Rightarrow$": Let $w_i := u_i(\sigma^*)$.

Clearly, every $i \in N$ and $s_i \in S_i$ satisfy $u_i(s_i, \sigma_{-i}^*) \leq u_i(\sigma^*) = w_i$.

By Corollary 24, there is *at least one* $s_i \in supp(\sigma_i^*)$ satisfying $u_i(s_i, \sigma_{-i}^*) = u_i(\sigma^*) = w_i$.

Now if there is $s_i' \in supp(\sigma_i^*)$ such that

$$u_i(s_i', \sigma_{-i}^*) < u_i(\sigma^*) \quad (= u_i(s_i, \sigma_{-i}^*))$$

then increasing the probability $\sigma_i^*(s_i)$ and decreasing (in proportion) $\sigma_i^*(s_i')$ strictly increases of $u_i(\sigma^*)$, a contradiction with σ^* being NE.

Example: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Player 1 (row) plays $(p(H), (1-p)(T))$ (we write just p) and player 2 (column) plays $(q(H), (1-q)(T))$ (we write q).

Compute all Nash equilibria.

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There are no equilibria where only player 1 randomizes:

Indeed, assume that (p, H) is such an equilibrium. Then by Lemma 42,

$$1 = u_1(H, H) = u_1(T, H) = -1$$

a contradiction. Also, (p, T) cannot be an equilibrium.

Similarly, there is no NE where only player 2 randomizes.

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Assume that both players randomize, i.e., $p, q \in (0, 1)$.

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Compute all Nash equilibria.

Assume that both players randomize, i.e., $p, q \in (0, 1)$.

The expected payoffs of playing pure strategies for player 1:

$$u_1(H, q) = 2q - 1 \text{ and } u_1(T, q) = 1 - 2q$$

Similarly for player 2 :

$$u_2(p, H) = 1 - 2p \text{ and } u_1(p, T) = 2p - 1$$

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Similarly for player 2 :

$$u_2(p, H) = 1 - 2p \text{ and } u_1(p, T) = 2p - 1$$

By Lemma 42, Nash equilibria must satisfy:

$$2q - 1 = 1 - 2q \qquad \text{and} \qquad 1 - 2p = 2p - 1$$

That is $p = q = \frac{1}{2}$ is the only Nash equilibrium.

Example: Battle of Sexes

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Player 1 (row) plays $(p(O), (1-p)(F))$ (we write just p) and player 2 (column) plays $(q(O), (1-q)(F))$ (we write q).

Compute all Nash equilibria.

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Compute all Nash equilibria.

There are two pure strategy equilibria $(2, 1)$ and $(1, 2)$, no Nash equilibrium where only one player randomizes.

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Compute all Nash equilibria.

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Now assume that

- player 1 (row) plays $(p(H),(1-p)(T))$ (we write just p) and
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where $p,q \in (0,1)$.

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where $p,q \in (0,1)$.

By Lemma 42, any Nash equilibrium must satisfy:

$$2q = 1 - q \qquad \text{and} \qquad p = 2(1-p)$$

This holds only for $q = \frac{1}{3}$ and $p = \frac{2}{3}$.

An Algorithm?

What did we do in the previous examples?

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Whenever one of the *supports* was non-singleton, we reduced computation of Nash equilibria to *linear equations*.

Support Enumeration (Idea)

Recall Lemma 42: $\sigma^* = (\sigma^*_1, \ldots, \sigma^*_n) \in \Sigma$ is a Nash equilibrium **iff** there exist $w_1, \ldots, w_n \in \mathbb{R}$ such that the following holds:

- For all $i \in N$ and all $s_i \in supp(\sigma^*_i)$ we have $u_i(s_i, \sigma^*_{-i}) = w_i$.
- For all $i \in N$ and all $s_i \notin supp(\sigma^*_i)$ we have $u_i(s_i, \sigma^*_{-i}) \leq w_i$.

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Suppose that we somehow know the supports $supp(\sigma^*_1), \ldots, supp(\sigma^*_n)$ for some Nash equilibrium $\sigma^*_1, \ldots, \sigma^*_n$ (which itself is unknown to us).

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Suppose that we somehow know the supports $supp(\sigma_1^*), \ldots, supp(\sigma_n^*)$ for some Nash equilibrium $\sigma_1^*, \ldots, \sigma_n^*$ (which itself is unknown to us).

Now we may consider all $\sigma_i^*(s_i)$'s and all w_i's as variables and use the above conditions to design a system of inequalities capturing Nash equilibria with the given support sets $supp(\sigma_1^*), \ldots, supp(\sigma_n^*)$.

Support Enumeration

To simplify notation, assume that for every i we have $S_i = \{1, \ldots, m_i\}$.
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$\sigma_1(1), \ldots, \sigma_1(m_1), \ldots, \sigma_n(1), \ldots, \sigma_n(m_n), w_1, \ldots, w_n$:

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$\sigma_1(1), \ldots, \sigma_1(m_1), \ldots, \sigma_n(1), \ldots, \sigma_n(m_n), w_1, \ldots, w_n$:

1. For all $i \in N$ and all $k \in supp_i$ we have

$$(u_i(k, \sigma_{-i}) =) \quad \sum_{s \in S \wedge s_i = k} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s) = w_i$$

2. For all $i \in N$ and all $k \notin supp_i$ we have

$$(u_i(k, \sigma_{-i}) =) \quad \sum_{s \in S \wedge s_i = k} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s) \leq w_i$$

3. For all $i \in N$: $\sigma_i(1) + \cdots + \sigma_i(m_i) = 1$.
4. For all $i \in N$ and all $k \in supp_i$: $\sigma_i(k) \geq 0$.
5. For all $i \in N$ and all $k \notin supp_i$: $\sigma_i(k) = 0$.

Support Enumeration

Consider the system of constraints from the previous slide.

The following lemma follows immediately from Lemma 42.

Lemma 43
Let $\sigma^* \in \Sigma$ be a strategy profile.

- *If σ^* is a Nash equilibrium and $supp(\sigma_i^*) = supp_i$ for all $i \in N$, then assigning $\sigma_i(k) := \sigma_i^*(k)$ and $w_i := u_i(\sigma^*)$ solves the system.*
- *If $\sigma_i(k) := \sigma_i^*(k)$ and $w_i := u_i(\sigma^*)$ solves the system, then σ^* is a Nash equilibrium with $supp(\sigma_i^*) \subseteq supp_i$ for all $i \in N$.*

Support Enumeration (Two Players)

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Input: A two-player strategic-form game G with strategy sets $S_1 = \{1, \ldots, m_1\}$ and $S_2 = \{1, \ldots, m_2\}$ and rational payoffs u_1, u_2.

Output: A Nash equilibrium σ^*.

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Algorithm: For all possible $supp_1 \subseteq S_1$ and $supp_2 \subseteq S_2$:

- Check if the corresponding system of linear constraints (from the previous slide) has a feasible solution $\sigma^*, w_1^*, \dots, w_n^*$.
- If so, STOP: the feasible solution σ^* is a Nash equilibrium satisfying $u_i(\sigma^*) = w_i^*$.

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Question: How many possible subsets $supp_1, supp_2$ are there to try?

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Question: How many possible subsets $supp_1, supp_2$ are there to try?
Answer: $2^{(m_1+m_2)}$

So, unfortunately, the algorithm requires worst-case exponential time.

Remarks on Support Enumeration

- The algorithm combined with Theorem 39 and properties of linear programming imply that every finite two-player game has a rational Nash equilibrium (furthermore, the rational numbers have polynomial representation in binary).

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  (There are algorithms for computing (a finite representation of) a set of all feasible solutions of a given linear constraint system.)
- The algorithm can be used to compute "good" equilibria.

  For example, to find a Nash equilibrium maximizing the sum of all expected payoffs (the "social welfare") it suffices to solve the system of constraints while maximizing $w_1 + \cdots + w_n$. More precisely, the algorithm can be modified as follows:
  - Initialize $W := -\infty$ (W stores the current maximum welfare)
  - For all possible $supp_1 \subseteq S_1$ and $supp_2 \subseteq S_2$:
    - Find the maximum value $\max(\sum w_i)$ of $w_1 + \cdots + w_n$ so that the constraints are satisfiable (using linear programming).
    - Put $W := \max\{W, \max(\sum w_i)\}$.
  - Return W.

Remarks on Support Enumeration (Cont.)

Similar trick works for any notion of "good" NE that can be expressed using a linear objective function and (additional) linear constraints in variables $\sigma_i(j)$ and w_i.

(e.g., maximize payoff of player 1, minimize payoff of player 2 and keep probability of playing the strategy 1 below 1/2, etc.)

Complexity Results – (Two Players)

Theorem 44

All the following problems are NP-complete: Given a two-player game in strategic form, does it have

1. *a NE in which player 1 has utility at least a given amount v ?*
2. *a NE in which the sum of expected payoffs of the two players is at least a given amount v ?*
3. *a NE with a support of size greater than a given number?*
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Membership to NP follows from the support enumeration:
For example, for 1., it suffices to guess supports $supp_1, supp_2$ and add $w_1 \geq v$ to the constraints; the resulting NE σ^* satisfies $u_1(\sigma^*) \geq v$.

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NP-hardness can be proved using reduction from SAT
(The reduction is not difficult but we are not going into it.

It is presented in "New Complexity Results about Nash Equilibria" by

V. Conitzer and T. Sandholm (pages 6–8))

The Reduction (It's Short and Sweet)

Definition 4 *Let ϕ be a Boolean formula in conjunctive normal form (representing a SAT instance). Let V be its set of variables (with $|V| = n$), L the set of corresponding literals (a positive and a negative one for each variable[6]), and C its set of clauses. The function $v : L \to V$ gives the variable corresponding to a literal,* e.g., *$v(x_1) = v(-x_1) = x_1$. We define $G_\epsilon(\phi)$ to be the following finite symmetric 2-player game in normal form. Let $\Sigma = \Sigma_1 = \Sigma_2 = L \cup V \cup C \cup \{f\}$. Let the utility functions be*

- $u_1(l^1, l^2) = u_2(l^2, l^1) = n - 1$ *for all* $l^1, l^2 \in L$ *with* $l^1 \neq -l^2$;
- $u_1(l, -l) = u_2(-l, l) = n - 4$ *for all* $l \in L$;
- $u_1(l, x) = u_2(x, l) = n - 4$ *for all* $l \in L$, $x \in \Sigma - L - \{f\}$;
- $u_1(v, l) = u_2(l, v) = n$ *for all* $v \in V$, $l \in L$ *with* $v(l) \neq v$;
- $u_1(v, l) = u_2(l, v) = 0$ *for all* $v \in V$, $l \in L$ *with* $v(l) = v$;
- $u_1(v, x) = u_2(x, v) = n - 4$ *for all* $v \in V$, $x \in \Sigma - L - \{f\}$;
- $u_1(c, l) = u_2(l, c) = n$ *for all* $c \in C$, $l \in L$ *with* $l \notin c$;
- $u_1(c, l) = u_2(l, c) = 0$ *for all* $c \in C$, $l \in L$ *with* $l \in c$;
- $u_1(c, x) = u_2(x, c) = n - 4$ *for all* $c \in C$, $x \in \Sigma - L - \{f\}$;
- $u_1(x, f) = u_2(f, x) = 0$ *for all* $x \in \Sigma - \{f\}$;
- $u_1(f, f) = u_2(f, f) = \epsilon$;
- $u_1(f, x) = u_2(x, f) = n - 1$ *for all* $x \in \Sigma - \{f\}$.

Theorem 1 *If $(l_1, l_2, \ldots, l_n)$ (where $v(l_i) = x_i$) satisfies ϕ, then there is a Nash equilibrium of $G_\epsilon(\phi)$ where both players play l_i with probability $\frac{1}{n}$, with expected utility $n - 1$ for each player. The only other Nash equilibrium is the one where both players play f, and receive expected utility ϵ each.*

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In what follows we show that

- the sample equilibrium problem can be solved in polynomial time for zero-sum two-player games,

  (Using a beautiful characterization of all Nash equilibria)
- the sample equilibrium problem belongs to the complexity class PPAD (which is a subclass of FNP) for two-player games.

  (... to be defined later)

Is there a better characterization of Nash equilibria than Lemma 42 ?

MaxMin

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Definition 46
$\sigma_i^* \in \Sigma_i$ is a *maxmin* strategy of player i if

$$\sigma_i^* \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}} \min_{\sigma_{-i} \in \Sigma_{-i}} u_i(\sigma_i, \sigma_{-i})$$

(Intuitively, a *maxmin* strategy σ_1^* maximizes player 1's worst-case payoff in the situation where player 2 strives to cause the greatest harm to player 1.)

(Since u_i is continuous and Σ_{-i} compact, $\min_{\sigma_{-i} \in \Sigma_{-i}} u_i(\sigma_i, \sigma_{-i})$ is well defined and continuous on Σ_i, which implies that there is at least one maxmin strategy.)

MaxMin

Lemma 47
σ_i^* *is maxmin iff*

$$\sigma_i^* \in \operatorname*{argmax}_{\sigma_i \in \Sigma_i} \min_{s_{-i} \in S_{-i}} u_i(\sigma_i, s_{-i})$$

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Lemma 47
σ_i^* is maxmin iff

$$\sigma_i^* \in \operatorname*{argmax}_{\sigma_i \in \Sigma_i} \min_{s_{-i} \in S_{-i}} u_i(\sigma_i, s_{-i})$$

Proof.
By Corollary 24, for every $\sigma \in \Sigma$ we have $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i, s_{-i})$ for some $s_{-i} \in S_{-i}$.

MaxMin

Lemma 47
σ_i^* is maxmin iff

$$\sigma_i^* \in \operatorname*{argmax}_{\sigma_i \in \Sigma_i} \min_{s_{-i} \in S_{-i}} u_i(\sigma_i, s_{-i})$$

Proof.
By Corollary 24, for every $\sigma \in \Sigma$ we have $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i, s_{-i})$ for some $s_{-i} \in S_{-i}$.

Thus $\min_{\sigma_{-i} \in \Sigma_{-i}} u_i(\sigma_i, \sigma_{-i}) = \min_{s_{-i} \in S_{-i}} u_i(\sigma_i, s_{-i})$.

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Question: Assume a strategy profile where both players play their maxmin strategies? Does it have to be a Nash equilibrium?

Zero-Sum Games: von Neumann's Theorem

Assume that G is zero sum, i.e., $u_1 = -u_2$.

Then $\sigma_2^* \in \Sigma_2$ is maxmin of player 2 **iff**

$$\sigma_2^* \in \operatorname*{argmin}_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2) \quad \left(= \operatorname*{argmin}_{\sigma_2 \in \Sigma_2} \max_{s_1 \in S_1} u_1(s_1, \sigma_2)\right)$$

(Intuitively, maxmin of player 2 minimizes the payoff of player 1 when player 1 plays his best responses. Such strategy of player 2 is often called minmax.)

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Theorem 48 (von Neumann)

Assume a two-player **zero-sum** *game. Then*

$$\max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2) = \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2)$$

Morever, $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ is a Nash equilibrium **iff** *both σ_1^* and σ_2^* are maxmin.*

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Morever, $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$ *is a Nash equilibrium* **iff** *both* σ_1^* *and* σ_2^* *are maxmin.*

So to compute a Nash equilibrium it suffices to compute (arbitrary) maxmin strategies for both players.

Proof of Theorem 48 (Homework)

Homework: Prove von Neumann's Theorem in 4 easy steps:

1. Prove this inequality:

$$\max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2) \leq \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2)$$

2. Prove that (σ_1^*, σ_2^*) is a Nash equilibrium iff

$$\min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1^*, \sigma_2) \geq u_1(\sigma_1^*, \sigma_2^*) \geq \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2^*)$$

   Hint: One of the inequalities is trivial and the other one almost.

3. Use 1. and 2. together with Theorem 39 to prove

$$\max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2) \geq \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2)$$

4. Use the above to prove the rest of the theorem.
   Hint: Use the characterization of NE from 2., do not forget that you already have $\max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2) = \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2)$
   You may already have proved one of the implications when proving 3.

Zero-Sum Two-Player Games – Computing NE

Assume $S_1 = \{1, \ldots, m_1\}$ and $S_2 = \{1, \ldots, m_2\}$.

Zero-Sum Two-Player Games – Computing NE

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We want to compute

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Consider a linear program with variables $\sigma_1(1), \ldots, \sigma_1(m_1), v$:

$$\begin{aligned}
\textbf{maximize:} \quad & v \\
\textbf{subject to:} \quad & \sum_{k=1}^{m_1} \sigma_1(k) \cdot u_1(k, \ell) \geq v & \ell = 1, \ldots, m_2 \\
& \sum_{k=1}^{m_1} \sigma_1(k) = 1 \\
& \sigma_1(k) \geq 0 & k = 1, \ldots, m_1
\end{aligned}$$

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Lemma 49

$\sigma_1^* \in \operatorname{argmax}_{\sigma_1 \in \Sigma_1} \min_{\ell \in S_2} u_1(\sigma_1, \ell)$ **iff** *assigning* $\sigma_1(k) := \sigma_1^*(k)$ *and* $v := \min_{\ell \in S_2} u_1(\sigma_1^*, \ell)$ *gives an optimal solution.*

Zero-Sum Two-Player Games – Computing NE

Summary:

- We have reduced computation of NE to computation of maxmin strategies for both players.
- Maxmin strategies can be computed using linear programming in polynomial time.
- That is, Nash equilibria in zero-sum two-player games can be computed in polynomial time.