Numerical methods

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Autumn 2017

Literature

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- Mathews, J.H., Fink, K.D.: Numerical methods using MATLAB, Pearson Prentice Hall, 2003
- Stoer, J., Bulirsch R.: Introduction to Numerical Analysis, Spriger, 1992

Assumptions

- Linear algebra
- Differential calculus
- Integral calculus

Errors

x: exact value, $\tilde{x}$: approximation x
$\tilde{x} - x$: *absolute error* $\tilde{x}$ $\quad |\tilde{x} - x| \leq \alpha$: *estimate of the absolute error*

$\frac{x-\tilde{x}}{x}$: *relative error* $\qquad \left|\frac{x-\tilde{x}}{x}\right| \leq \delta$: *estimate of the relative error*

Approximation $\tilde{x}$ of x to s digits:

$$\left|\frac{x - \tilde{x}}{x}\right| \leq 5.10^{-s}.$$

Error for the computer representation of the numbers

Matlab code:

```
>> c=0.1;
>> x=c*ones(1,100);
>> sum(x)-10
ans =
    -1.953992523340276e-14

% rearrangement of the computation

>> A=reshape(x,10,10);
>> sum(sum(A))-10
ans =
    -1.776356839400250e-15
```

Conditionality

Condition number:

$$C_p = \frac{\left|\dfrac{\Delta y}{y}\right|}{\left|\dfrac{\Delta x}{x}\right|} = \frac{|\text{output relative error}|}{|\text{input relative error}|}$$

$C_p \approx 1$ – the task is *well-conditioned*
$C_p > 100$ – the task is *ill-conditioned*

Condition number for matrix

A – non-singular matrix
$C_A = \|A\| \cdot \|A^{-1}\|$

Example of ill-conditioned matrix:
Hilbert matrix $H = (h_{i,j})$, $h_{i,j} = \frac{1}{i+j-1}$

Matlab code:

```
>> H=hilb(8)
H =
 1      1/2    1/3    1/4    1/5    1/6    1/7    1/8
 1/2    1/3    1/4    1/5    1/6    1/7    1/8    1/9
 1/3    1/4    1/5    1/6    1/7    1/8    1/9    1/10
 1/4    1/5    1/6    1/7    1/8    1/9    1/10   1/11
 1/5    1/6    1/7    1/8    1/9    1/10   1/11   1/12
 1/6    1/7    1/8    1/9    1/10   1/11   1/12   1/13
 1/7    1/8    1/9    1/10   1/11   1/12   1/13   1/14
 1/8    1/9    1/10   1/11   1/12   1/13   1/14   1/15
>> cond(H)
ans =
15257575253
```

Typical ill-conditioned tasks

- dividing by a small number
- subtracting almost the same numbers
- cumulation of errors in iterative calculation

$$A_n = n \cdot A_{n-1}$$

Example:
$I_n = \int_0^1 x^n e^{x-1} dx$
$I_1 = \frac{1}{1}$, itegration by parts: $I_n = 1 - nI_{n-1}$
The initial error is multiplied by n in every step, i.e. for $n = 10$ is the error multiplied by $10! = 3,628,800$.

Symbols O, o

f, g – functions defined in the neighbourhood of a point a (it is possible $a = \pm\infty$)

$f(x) = O(g(x))$ for $x \to a$ there exists a constant $C > 0$:

$$|f(x)| \leq C \cdot |g(x)|$$

in the neigbourhood of a point a.

Meaning:
function f is similar to g in the neigbourhood of a point a.

$$f(x) = o(g(x)) \text{ for } x \to a \qquad \lim_{x \to a} \frac{|f(x)|}{|g(x)|} = 0\,.$$

Meaning:
function f converges to 0 faster then g in the point a.

For sequences $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$:

$$a_n = O(b_n) \text{ or } a_n = o(b_n) \text{ pro } n \to \infty\,,$$

Sometimes the point a is clear – we can omit it:

$$a_n = O(1/n), \qquad f(h) = o(h^3)$$

Example: Taylor series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(\xi)}{2}h^2$$

$$f(x+h) = f(x) + f'(x)h + O(h^2), \quad f(x+h) = f(x) + f'(x)h + o(h)$$