Numerical methods – lecture 2

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Solving nonlinear equations

Equation

$$f(x) = 0,$$

$x \in I = [a, b]$, f is continuous real function

$\hat{x} \in I$ – solution, root of f.

Iterative process: We create sequence $(x_k)_{k=0}^{\infty}$, $x_k \to \hat{x}$.
$(x_k)_{k=0}^{\infty}$ – iterative sequence.

Bisection method

$f(a) \cdot f(b) \leq 0$, $a_0 = a$, $b_0 = b$, let $x_0 = (a_0 + b_0)/2$.

If $f(a_0) \cdot f(x_0) \leq 0$ we choose
$a_1 = a_0,\ b_1 = x_0$, else
$a_1 = x_0,\ b_1 = b_0$,
$\hat{x} \in [a_1, b_1]$.

Generally: we have a_k, b_k, $f(a_k) \cdot f(b_k) \leq 0$, $\hat{x} \in [a_k, b_k]$, let $x_k = (a_k + b_k)/2$.

If $f(a_k) \cdot f(x_k) \leq 0$ we choose
$a_{k+1} = a_k$, $b_{k+1} = x_k$, else
$a_{k+1} = x_k$, $b_{k+1} = b_k$,
so $\hat{x} \in [a_{k+1}, b_{k+1}]$.

Estimate of the absolute error in k-th step:

$$|x_k - \hat{x}| \leq \frac{b-a}{2^{k+1}}$$

Fixed point iteration

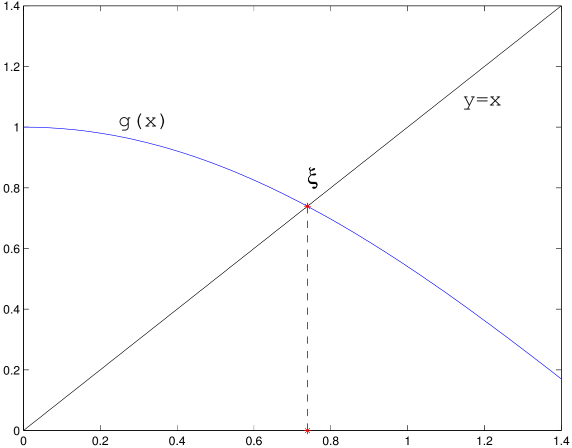
- Equation
$$x = g(x)$$
- g continuous on $I = [a, b]$
- Solution $\hat{x}$ is called the **fixed point** of the function g

Iteration process

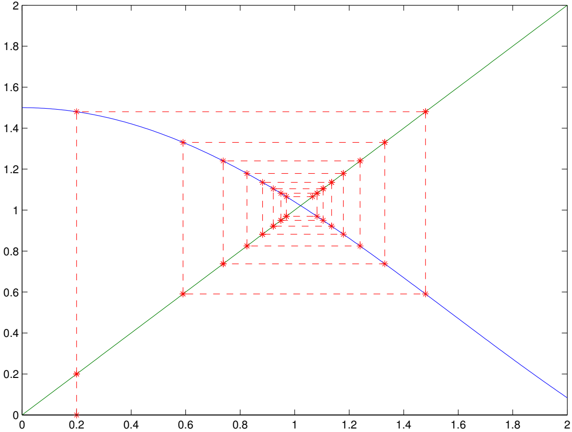
- Let us choose $x_0 \in I$ and $x_1 = g(x_0)$.
- Generally $x_{k+1} = g(x_k)$.
- Function g is called **iteration function**.

Geometric meaning

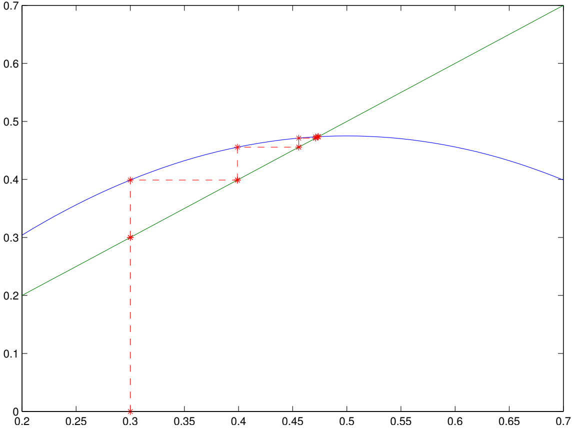
The fixed point $\hat{x}$ is the intersection of the function g and line $y = x$.



Graphical representation of the iteration process:



The convergence is faster if the derivative of g in the intersection is close to 0:



The existence and uniqueness of the fixed point

Theorem: If for the function g continuous on $I = [a, b]$ the following condition holds

$$\forall x \in I : g(x) \in I,$$

then there exists at least one fixed point $\hat{x} \in I$ of the function g. Moreover, if there exits constant $L < 1$ that for all $x \in I$

$$\forall x \in I : |g'(x)| \leq L,$$

then there exit one fixed point $\hat{x}$ and for any $x_0 \in I$ the iteration process given by formula

$$x_{k+1} = g(x_k)$$

converges to this fixed point.

Estimation of the error

$$|x_k - \xi| \leq \frac{L^k}{1-L} |x_0 - x_1|$$

Example

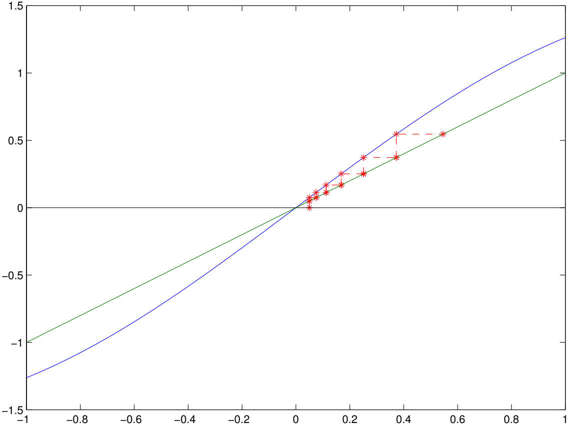
$$x^3 + 4x^2 - 10 = 0$$

Classification of the fixed points

The fixed point $\hat{x}$ of the function g is called

- **attractive** if $|g'(\hat{x})| < 1$, then the iterative process converges on some neighborhood of $\hat{x}$.
- **repelling** if $|g'(\hat{x})| > 1$, then the iterative process doesn't converge.

the process doesn't converge if $|g'(\hat{x})| > 1$:



Creating of the iteration function

$$f(x) = 0 \qquad \to x = g(x)$$

$$g(x) = x - \frac{f(x)}{K}$$

Generally:

$$g(x) = x - \frac{f(x)}{h(x)}$$