Numerical methods – lecture 3

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Repetition

Equation

$$f(x) = 0 \qquad \longmapsto \qquad x = g(x)$$

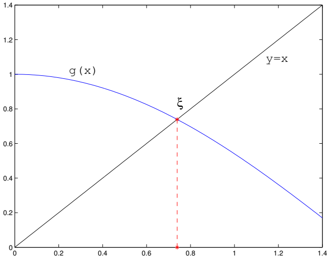
Fixed point method:

Iteration process:

$$x_{k+1} = g(x_k)$$

Geometric meaning

The fixed point $\hat{x}$ is the intersection of the function g and line $y = x$.



The existence and uniqueness of the fixed point

Theorem: If for the function g continuous on $I = [a, b]$ the following condition holds

$$\forall x \in I : g(x) \in I,$$

then there exists at least one fixed point $\hat{x} \in I$ of the function g. Moreover, if there exits constant $L < 1$ that for all $x \in I$

$$\forall x \in I : |g'(x)| \leq L,$$

then there exit one fixed point $\hat{x}$ and for any $x_0 \in I$ the iteration process given by formula

$$x_{k+1} = g(x_k)$$

converges to this fixed point.

Function g is called **contraction**.

The error of the iteration:

$$|x_k - \xi| \le \frac{L^k}{1-L} |x_0 - x_1|$$

Classification of the fixed points

The fixed point $\hat{x}$ of the function g is called

- **attractive** if $|g'(\hat{x})| < 1$, then the iterative process converges on some neighborhood of $\hat{x}$.
- **repelling** if $|g'(\hat{x})| > 1$, then the iterative process doesn't converge.

Creating of the iteration function

$$g(x) = x - \frac{f(x)}{K}$$

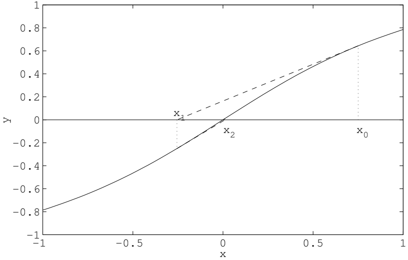
$$g(x) = x - \frac{f(x)}{h(x)}$$

Newton(–Raphson) method

Let us return to the equation

$$f(x) = 0.$$

x_0 – initial iteration, x_1 – intersection of the tangent to f in x_0 the axis x.



$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Iteration function:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

x_{k+1} – intersection of the tangent to f in x_k the axis x
$\rightarrow$ **tangent method**

Theorem 1

Newtom methods converges to the root $\hat{x}$ if the function f has continuous derivative in some neighborhood of $\hat{x}$, $f'(\hat{x}) \neq 0$ and the initial iteration x_0 is close enogh to $\hat{x}$.

Theorem 2

If f has continuous the second derivative in some neighborhood of $\hat{x}$ and $f'(\hat{x}) \neq 0$ then $g'(\hat{x}) = 0$ for the iteration function of Newton method.

Example 1:

Computation of $\sqrt(a)$

$f(x) = x^2 - a$, $f'(x) = 2x$.

$$x_{k+1} = x_k - \frac{x_k^2 - a}{2x_k} = \frac{x_k^2 + a}{2x_k}$$

Example :

Computation of $\frac{1}{a}$ without division:

$f(x) = \frac{1}{x} - a$

$$x_{k+1} = x_k(2 - ax_k)$$

Fourier conditions

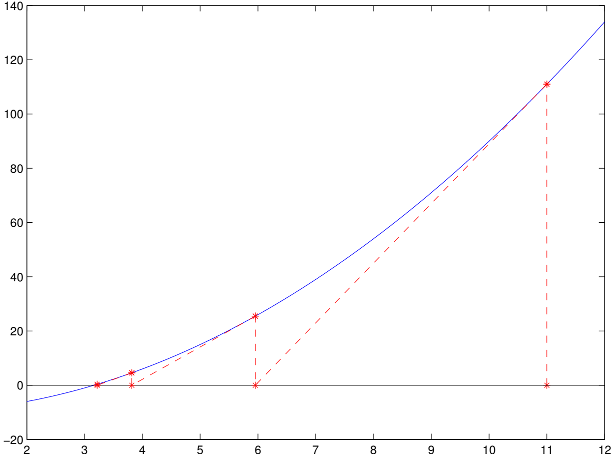
Theorem 3

1. Let f has continuous the second derivative in $[a, b]$, $f(a) \cdot f(b) \leq 0$.
2. Let $\forall x \in [a, b] : f'(x) \neq 0$ and f'' doesn't change its sign in $[a, b]$

Let's choose $x_0 \in \{a, b\}$ such that $f(x_0) \cdot f'' \geq 0$. Then the sequence generated by Newton method converges monotonously to $\hat{x}$.

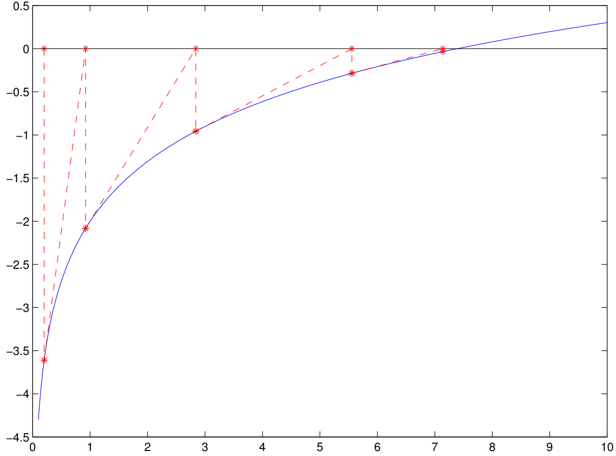
Fourier conditions for convex function

$f(x_0) > 0$



Fourier conditions for concave function

$f(x_0) < 0$

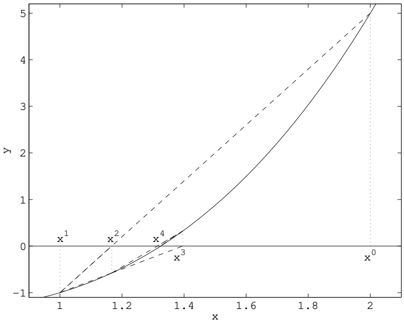


Methods derived from Newton method

Secant methods

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}, \qquad i = 1, 2, \ldots$$

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k), \qquad i = 1, 2, \ldots$$



Methods derived from Newton method

False position methods (regula falsi)

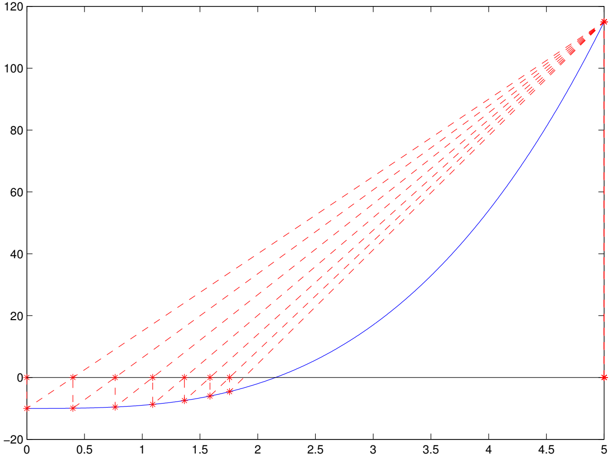
Similar to secant method with sign control: $f(a)f(b) < 0$, $f \in C[a,b]$, $x_0 = a$, $x_1 = b$, $f(x_0)f(x_1) < 0$

$$x_{k+1} = x_k - \frac{x_k - x_s}{f(x_k) - f(x_s)} f(x_k), \qquad k = 0, 1, \dots,$$

wher s is the largest index for which $f(x_k)f(x_s) < 0$.

Remark: If f is convex or concave in $[a,b]$ then $s = 0$ or $s = 1$ for all iterations.

Regula falsi for convex function



Order of the convefgence

Let $p \geq 1$, $x_k \to \hat{x}$, $e_k = x_k - \hat{x}$. If

$$\lim_{k \to \infty} \frac{|e_k|}{|e_{k+1}|^p} = C < \infty$$

then p is called the **order (rate)** of the convergence of the sequence $(x_k)_{k=0}^{\infty}$.

If the sequence $(x_k)_{k=0}^{\infty}$ is generated by the numerical methods, then p is the **order (rate) of the method**.

Theorem

Let the derivatives of the iteration function g be continuouns to order $q \geq p$. Then the order of the convergence of the sequence $(x_k)_{k=0}^{\infty}$ generated by the iteration process $x_{k+1} = g(x_k)$ is equal to p iff
$g(\hat{x}) = \hat{x}$, $g'(\hat{x}) = 0$, $g''(\hat{x}) = 0, \ldots, g^{(p-1)}(\hat{x}) = 0$,
$g^{(p)}(\hat{x}) \neq 0$,

Orders of methods:

Fixed point	1
Newton	2
Secant	$\frac{1+\sqrt{5}}{2} \doteq 1.618$
Regula falsi	1