Numerical methods – lecture 4

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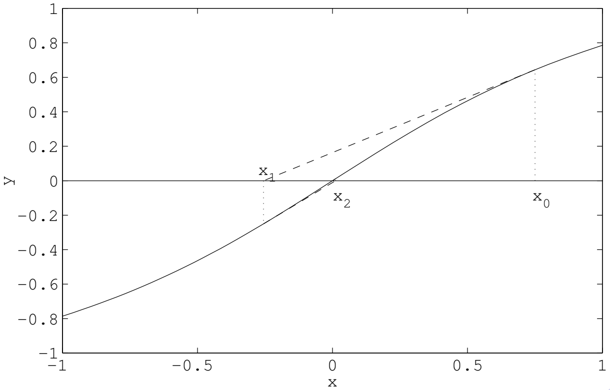
Newton method

$$f(x) = 0, \qquad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, 2, \ldots$$



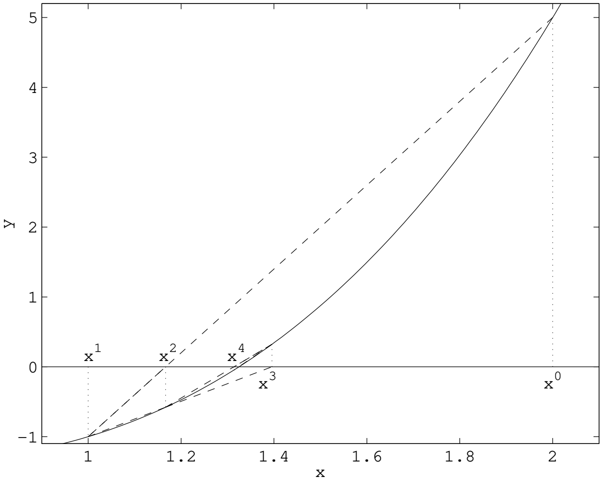
Fourier conditions

1. Let f has continuous the second derivative in $[a, b]$, $f(a) \cdot f(b) \leq 0$.
2. Let $\forall x \in [a, b] : f'(x) \neq 0$ and f'' doesn't change its sign in $[a, b]$

Let's choose $x_0 \in \{a, b\}$ such that $f(x_0) \cdot f'' \geq 0$. Then the sequence generated by Newton method converges monotonously to $\hat{x}$.

Secant methods

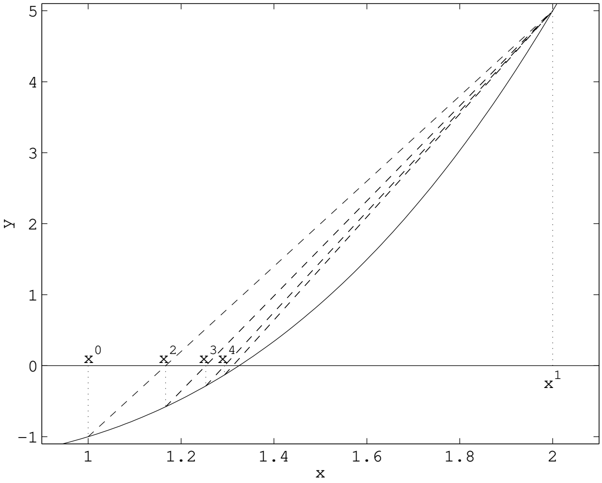
$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k), \qquad k = 1, 2, \ldots$$



Method regula falsi

$$x_{k+1} = x_k - \frac{x_k - x_s}{f(x_k) - f(x_s)} f(x_k), \qquad k = 1, 2, \ldots,$$

wher s is the largest index for which $f(x_k)f(x_s) \leq 0$.



Order of the convefgence

Let $p \geq 1$, $x_k \to \hat{x}$, $e_k = x_k - \hat{x}$. If

$$\lim_{k\to\infty} \frac{|e_k|}{|e_{k+1}|^p} = C < \infty$$

then p is called the **order (rate)** of the convergence of the sequence $(x_k)_{k=0}^{\infty}$.

If the sequence $(x_k)_{k=0}^{\infty}$ is generated by the numerical methods, then p is the **order (rate) of the method**.

$p = 1 \ \to$ **linear method**

$p = 2 \ \to$ **quadratic method**

Theorem

Let the derivatives of the iteration function g be continuouns to order $q \geq p$. Then the order of the convergence of the sequence $(x_k)_{k=0}^{\infty}$ generated by the iteration process $x_{k+1} = g(x_k)$ is equal to p iff
$g(\hat{x}) = \hat{x}$, $g'(\hat{x}) = 0$, $g''(\hat{x}) = 0, \ldots$, $g^{(p-1)}(\hat{x}) = 0$,
$g^{(p)}(\hat{x}) \neq 0$,

Orders of methods:

Fixed point	1
Newton	2
Secant	$\frac{1+\sqrt{5}}{2} \doteq 1.618$
Regula falsi	1

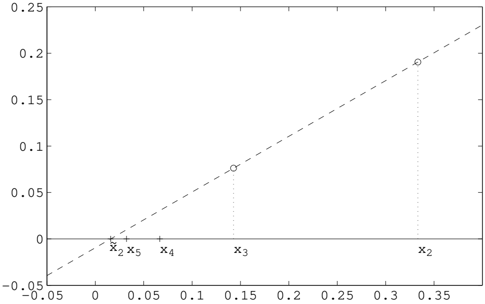
Example: geometric sequence

Geometric derivation

Let

$$\varepsilon(x_k) = x_k - x_{k+1}, \qquad \varepsilon(x_{k+1}) = x_{k+1} - x_{k+2}.$$

Points $[x_k, \varepsilon(x_k)]$, $[x_{k+1}, \varepsilon(x_{k+1})]$ are connected by the line. Its intersection with the axis x is the approximation of the limit of the sequence x_k.



The equation of the line:

$$y - \varepsilon(x_k) = \frac{\varepsilon(x_k) - \varepsilon(x_{k+1})}{x_k - x_{k+1}}(x - x_k)$$

The intersection with the axes x:

$$\tilde{x}_k = x_k - \frac{\varepsilon(x_k)(x_k - x_{k+1})}{\varepsilon(x_k) - \varepsilon(x_{k+1})} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}.$$

Theorem

Let $\{x_k\}_{k=0}^{\infty}$, $\lim_{k\to\infty} x_k = \hat{x}$, $x_k \neq \hat{x}$, $k = 0, 1, 2, \ldots$, be a sequence and let

$$x_{k+1} - \hat{x} = (C + \gamma_k)(x_k - \hat{x}),\ k = 0, 1, 2, \ldots,\ |C| < 1,\ \lim_{k\to\infty} \gamma_k = 0.$$

Then

$$\tilde{x}_k = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$$

is defined for k enough large and

$$\lim_{k\to\infty} \frac{\tilde{x}_k - \hat{x}}{x_k - \hat{x}} = 0,$$

i.e., the sequence $\{\tilde{x}_k\}$ converges to $\hat{x}$ faster than $\{x_k\}$.

Ordinary differences:

$\Delta x_k = x_{k+1} - x_k$

$\Delta^2 x_k = \Delta x_{k+1} - \Delta x_k = x_{k+2} - 2x_{k+1} + x_k$

$\Delta^3 x_k = \Delta^2 x_{k+1} - \Delta^2 x_k$

$\vdots$

$$\tilde{x}_k = x_k - \frac{(\Delta x_k)^2}{\Delta^2 x_k}$$

Steffensen method

Let g be iteration function for the equation $x = g(x)$. Let's put

$$y_k = g(x_k), \qquad z_k = g(y_k),$$
$$x_{k+1} = x_k - \frac{(y_k - x_k)^2}{z_k - 2y_k + x_k}.$$

This method id called **Steffensen method** and it can be described bz the iteration function φ:

$$x_{k+1} = \varphi(x_k),$$

for

$$\varphi(x) = x - \frac{(g(x) - x)^2}{g(g(x)) - 2g(x) + x} = \frac{xg(g(x)) - g^2(x)}{g(g(x)) - 2g(x) + x}.$$

Theorem

1. If $\varphi(\hat{x}) = \hat{x}$ then $g(\hat{x}) = \hat{x}$.
2. If $g(\hat{x}) = \hat{x}$, the derivative $g'(\hat{x})$ exits and $g'(\hat{x}) \neq 1$, then $\varphi(\hat{x}) = \hat{x}$.

Newton method

$$F(\boldsymbol{x}) = \boldsymbol{o}, \qquad F \in C^2(O(\boldsymbol{\xi}))$$

$$J_F(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_1(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial f_1(\boldsymbol{x})}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial f_m(\boldsymbol{x})}{\partial x_m} \end{pmatrix}$$

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - J_F^{-1}(\boldsymbol{x}^k) F(\boldsymbol{x}^k)$$

Iteration function

$$G(\boldsymbol{x}) = \boldsymbol{x} - J_F^{-1}(\boldsymbol{x}) F(\boldsymbol{x})$$