Numerical methods – lecture 5

Jiří Zelinka

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Iteration methods for solving system of linear equations

$$A\mathbf{x} = \mathbf{b} \quad \longrightarrow \quad \mathbf{x} = T\mathbf{x} + \mathbf{g}$$

Iteration process:

$$\mathbf{x}^{k+1} = T\mathbf{x}^{k} + \mathbf{g}, \qquad k = 0, 1, \ldots$$

Solution:

$$\hat{\mathbf{x}} = (E - T)^{-1} g$$

Theorem

The sequence $\left\{\mathbf{x}^k\right\}_{k=0}^{\infty}$ determined by the iteration process $\mathbf{x} = T\mathbf{x} + \mathbf{g}$ converges for every initial iteration $\mathbf{x}^0 \in \mathbb{R}^n \iff \rho(T) < 1$. In this case

$$\lim_{k\to\infty} \mathbf{x}^k = \hat{\mathbf{x}},\ \hat{\mathbf{x}} = T\hat{\mathbf{x}} + \mathbf{g}$$

Jacobi iteration method

System of linear equations:

$$A\mathbf{x} = \mathbf{b}$$

i-th equation:

$$a_{i1}x_1 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n = b_i$$

The component x_i is expressed

$$x_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a_{ij}}{a_{ii}} x_j + \frac{b_i}{a_{ii}},$$

and it is used as the new $(k+1)$-th iteration

$$x_i^{k+1} = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{a_{ij}}{a_{ii}} x_j^k + \frac{b_i}{a_{ii}},$$

Matrix notation

$$
\begin{pmatrix} x_1^{k+1} \\ x_2^{k+1} \\ \vdots \\ x_n^{k+1} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & & -\frac{a_{2n}}{a_{22}} \\ \vdots & & \ddots & \vdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1^{k} \\ x_2^{k} \\ \vdots \\ x_n^{k} \end{pmatrix} + \begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \vdots \\ \frac{b_n}{a_{nn}} \end{pmatrix} .
$$

$$Ax = \mathbf{b}, \qquad A = D + L + U,$$

$$Ax = (D + L + U)\mathbf{x} = \mathbf{b}$$

$$D = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & & & 0 \\ a_{21} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & 0 \end{pmatrix}.$$

$$\mathbf{x} = -D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b}.$$

$$\mathbf{x}^{k+1} = -D^{-1}(L+U)\mathbf{x}^{k} + D^{-1}\mathbf{b}.$$

$$\mathbf{x}^{k+1} = T_J \mathbf{x}^k + D^{-1}\mathbf{b},$$

$$T_J = -D^{-1}(L+U),\ t_{ij} = -\frac{a_{ij}}{a_{ii}} \text{ for } i \neq j,\ t_{ii} = 0.$$

$$T_J = \begin{pmatrix} 0 & -\dfrac{a_{12}}{a_{11}} & \cdots & -\dfrac{a_{1n}}{a_{11}} \\ -\dfrac{a_{21}}{a_{22}} & 0 & & -\dfrac{a_{2n}}{a_{22}} \\ \vdots & & \ddots & \vdots \\ -\dfrac{a_{n1}}{a_{nn}} & -\dfrac{a_{n2}}{a_{nn}} & \cdots & 0 \end{pmatrix}, \quad D^{-1}\mathbf{b} = \begin{pmatrix} \dfrac{b_1}{a_{11}} \\ \dfrac{b_2}{a_{22}} \\ \vdots \\ \dfrac{b_n}{a_{nn}} \end{pmatrix}.$$

Gauss–Seidel iteration method

The component of the new iteration is used in the following step:

$$
\begin{aligned}
x_1^{k+1} &= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^k - a_{13}x_3^k - a_{14}x_4^k - \ldots,\right) \\
x_2^{k+1} &= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{k+1} - a_{23}x_3^k - a_{24}x_4^k - \ldots,\right) \\
x_3^{k+1} &= \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{k+1} - a_{32}x_2^{k+1} - a_{34}x_4^k - \ldots,\right) \\
&\vdots \\
x_i^{k+1} &= \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^k\right)
\end{aligned}
$$

Matrix notation:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ (D+L+U)\mathbf{x} &= \mathbf{b} \\ (D+L)\mathbf{x} &= -U\mathbf{x}+\mathbf{b} \\ \mathbf{x} &= -(D+L)^{-1}U\mathbf{x}+(D+L)^{-1}\mathbf{b} \end{aligned}$$

$$T_G = -(D+L)^{-1}U, \qquad \mathbf{x}^{k+1} = T_G\mathbf{x}^k + (D+L)^{-1}\mathbf{b}.$$

Theorem: If A is diagonally dominant matrix, i.e.

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}| \quad \text{or} \quad |a_{ii}| > \sum_{j\neq i} |a_{ji}|$$

then Jacobi and Gauss–Seidel methods converge.

Relaxation (Succesive over–relaxation (SOR)) method

x^k – k-th iteration

x_{GS}^{k+1} – the following iteration aquired by the Gauss–Seidel metod

$\omega \in (0, 2)$ – relaxation parameter

$$x^{k+1} = (1 - \omega)x^k + \omega x_{GS}^{k+1}$$