Numerical methods – lecture 6

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Iteration methods for solving system of linear equations

$$A\mathbf{x} = \mathbf{b} \quad \longrightarrow \quad \mathbf{x} = T\mathbf{x} + \mathbf{g}$$

Iteration process:

$$\mathbf{x}^{k+1} = T\mathbf{x}^k + \mathbf{g}, \qquad k = 0, 1, \dots$$

Solution:

$$\hat{\mathbf{x}} = (E - T)^{-1} g$$

Theorem

The sequence $\left\{\mathbf{x}^k\right\}_{k=0}^{\infty}$ determined by the iteration process $\mathbf{x} = T\mathbf{x} + \mathbf{g}$ converges for every initial iteration $\mathbf{x}^0 \in \mathbb{R}^n \iff \rho(T) < 1$. In this case

$$\lim_{k\to\infty} \mathbf{x}^k = \hat{\mathbf{x}},\ \ \hat{\mathbf{x}} = T\hat{\mathbf{x}} + \mathbf{g}$$

Jacobi iteration method

i-th equation:

$$a_{i1}x_1 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n = b_i$$

The component x_i is expressed as k-th iteration:

$$x_i^{k+1} = -\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{a_{ij}}{a_{ii}} x_j^k + \frac{b_i}{a_{ii}},$$

Matrix notation

$$A\mathbf{x} = \mathbf{b}, \qquad A = D + L + U,$$

$$A\mathbf{x} = (D + L + U)\mathbf{x} = \mathbf{b}$$

$$D = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & & & 0 \\ a_{21} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & 0 \end{pmatrix}.$$

$$\mathbf{x} = -D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b}.$$

$$\mathbf{x}^{k+1} = -D^{-1}(L+U)\mathbf{x}^{k} + D^{-1}\mathbf{b}.$$

$$\mathbf{x}^{k+1} = T_J \mathbf{x}^k + D^{-1}\mathbf{b},$$

$$T_J = -D^{-1}(L+U),\ t_{ij} = -\frac{a_{ij}}{a_{ii}} \text{ for } i \neq j,\ t_{ii} = 0.$$

$$T_J = \begin{pmatrix} 0 & -\dfrac{a_{12}}{a_{11}} & \cdots & -\dfrac{a_{1n}}{a_{11}} \\ -\dfrac{a_{21}}{a_{22}} & 0 & & -\dfrac{a_{2n}}{a_{22}} \\ \vdots & & \ddots & \vdots \\ -\dfrac{a_{n1}}{a_{nn}} & -\dfrac{a_{n2}}{a_{nn}} & \cdots & 0 \end{pmatrix}, \quad D^{-1}\mathbf{b} = \begin{pmatrix} \dfrac{b_1}{a_{11}} \\ \dfrac{b_2}{a_{22}} \\ \vdots \\ \dfrac{b_n}{a_{nn}} \end{pmatrix}.$$

Gauss–Seidel iteration method

$$
\begin{aligned}
x_1^{k+1} &= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^k - a_{13}x_3^k - a_{14}x_4^k - \dots,\right) \\
x_2^{k+1} &= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{k+1} - a_{23}x_3^k - a_{24}x_4^k - \dots,\right) \\
x_3^{k+1} &= \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{k+1} - a_{32}x_2^{k+1} - a_{34}x_4^k - \dots,\right) \\
&\vdots \\
x_i^{k+1} &= \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^k\right)
\end{aligned}
$$

$$
\begin{aligned}
A\mathbf{x} &= \mathbf{b} \\
(D + L + U)\mathbf{x} &= \mathbf{b} \\
(D + L)\mathbf{x} &= -U\mathbf{x} + \mathbf{b} \\
\mathbf{x} &= -(D + L)^{-1}U\mathbf{x} + (D + L)^{-1}\mathbf{b}
\end{aligned}
$$

$$
T_G = (D + L)^{-1}U, \qquad \mathbf{x}^{k+1} = T_G\mathbf{x}^k + (D + L)^{-1}\mathbf{b}.
$$

Theorem: If A is diagonally dominant matrix, i.e.

$$
|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \text{or} \quad |a_{ii}| > \sum_{j \neq i} |a_{ji}|
$$

then Jacobi and Gauss–Seidel methods converge.

Relaxation (Succesive over–relaxation (SOR)) method

x^k – k-th iteration

x_{GS}^{k+1} – the following iteration aquired by the Gauss–Seidel metod

$\omega \in (0, 2)$ – relaxation parameter

$x^{k+1} = (1 - \omega)x^k + \omega x_{GS}^{k+1}$

Direct methods for solving system of linear equations

$A\mathbf{x} = \mathbf{b}$, A – non-singular

Gaussian elimination method:
reduction $[A|b]$ to the system with upper triangular matrix $[U|\tilde{b}]$.

Operations:

- Row exchange
- Adding c-multiple of the i-th row to the j-th row

Corresponding matrices:

$$
P_{i,k} = \begin{pmatrix}
1 & 0 & & & & & \cdots & & & & & 0 \\
0 & 1 & & & & & & & & & & 0 \\
 & & \ddots & & & & & & & & & \\
 & & & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & & & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
 & & & 0 & 0 & 1 & & & 0 & 0 & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & \ddots & & \vdots & \vdots & & \vdots \\
 & & & 0 & 0 & & & 1 & 0 & 0 & \cdots & 0 \\
 & & & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & & & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & & \\
 & & & \vdots & & & & & & & \ddots & \\
0 & & & 0 & & & & & & & & 1
\end{pmatrix}
\begin{matrix} \\ \\ \\ \\ i \\ \\ \\ \\ k \\ \\ \\ \\ \end{matrix}
$$

$P_{i,k}$ – permutation matrixm $P_{i,k}^{-1} = P_{i,k}^{T} = P_{i,k}$.

$$
L_{i,k,c} = \left(\begin{array}{cccccccc}
1 & 0 & & \cdots & & & & 0 \\
0 & 1 & & \cdots & & & & 0 \\
\vdots & & \ddots & & & & & \\
 & & & 1 & & & & \\
 & & & & \ddots & & & \\
 & & & c & & 1 & & \\
 & & & & & & \ddots & \\
0 & & & & & & & 1
\end{array}\right)
\begin{array}{c}
\\ \\ \\ i \\ \\ k \\ \\ \\
\end{array}
$$

$$
L_{i,k,c}^{-1} = \begin{pmatrix}
1 & 0 & & \cdots & & & 0 \\
0 & 1 & & \cdots & & & 0 \\
\vdots & & \ddots & & & & \\
 & & & 1 & & & \\
 & & & & \ddots & & \\
 & & & -c & & 1 & \\
 & & & & & & \ddots \\
0 & & & & & & 1
\end{pmatrix}
\begin{matrix}
\\ \\ \\ i \\ \\ k \\ \\ \\
\end{matrix}
$$

Gaussian elimination method without row exchange

(1) zeroing the first column under the diagonal

$$\left(A^{(1)} \mid \mathbf{b}^{(1)}\right) = L_1 \cdot (A \mid \mathbf{b})\,, \quad L_1 = \begin{pmatrix} 1 & \cdots & \cdots & 0 \\ l_{21} & 1 & & \\ \vdots & & \ddots & \vdots \\ l_{n1} & 0 & \cdots & 1 \end{pmatrix},$$

$$l_{k1} = -\frac{a_{k1}}{a_{11}}$$

(i) zeroing the i-th column under diagonal

$$\left(A^{(i)} \mid \mathbf{b}^{(i)}\right) = L_i \cdot \left(A^{(i-1)} \mid \mathbf{b}^{(i-1)}\right),$$

$$L_i = \begin{pmatrix} 1 & & \cdots & & 0 \\ & \ddots & & & \\ \vdots & & 1 & & \\ & & l_{i+1,i} & \ddots & \\ & & \vdots & & \ddots \\ 0 & & l_{n,i} & & 1 \end{pmatrix}$$

$$l_{ki} = -\frac{a_{ki}^{(i-1)}}{a_{ii}^{(i-1)}}, \quad i = 2, \ldots, n-1$$

$$\left(U \mid \tilde{\mathbf{b}}\right) = L_{n-1} \cdot \ldots L_2 \cdot L_1 \cdot (A \mid \mathbf{b})$$

so

$$U = L_{n-1} \cdot \ldots L_2 \cdot L_1 \cdot A$$

then

$$L_1^{-1} L_2^{-1} \ldots L_{n-1}^{-1} U = A.$$

Matrices L_i are lower triangular so matrices L_1^{-i} are lower triangular, too. Then

$$A = L \cdot U, \qquad L = L_1^{-1} L_2^{-1} \ldots L_{n-1}^{-1}.$$

L – lower triangular matrix.

$$L_i^{-1} = \begin{pmatrix} 1 & & \cdots & & & 0 \\ & \ddots & & & & \\ \vdots & & 1 & & & \\ & & -l_{i+1,i} & \ddots & & \\ & & \vdots & & \ddots & \\ 0 & & -l_{ni} & & & 1 \end{pmatrix},$$

$$L = L_1^{-1} L_2^{-1} \dots L_{n-1}^{-1} = \begin{pmatrix} 1 & \cdots & \cdots & 0 \\ -l_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -l_{n1} & \cdots & -l_{n,n-1} & 1 \end{pmatrix}.$$

LU decomposition

The product $A = L \cdot U$ is called LU (LR) decomposition of the matrix A

Solving linear equations:
substitution $U\mathbf{x} = \mathbf{y}$, and we get system $L\mathbf{y} = \mathbf{b}$ with the lower triangular matrix, then we solve $U\mathbf{x} = \mathbf{y}$ with the upper triangular matrix.

LU decomposition with row exchange

$$P \cdot A = L \cdot U$$

for suitable permutation matrix P.

We must do row exchange if $a_{ii} = 0$.
Pradcically, we find out the row exchanges during calculation.

Pivoting (partial)

To improve the numerical stability the element with maximal absolut value is chosen in each column in the rest of the matrix to be modify. Then we exchange appropriate rows.

Example

$$\begin{aligned} 2x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \\ 4x_1 + x_2 + 2x_3 &= 9 \end{aligned}$$

$$A = \begin{pmatrix} 2 & 4 & -1 \\ 1 & 1 & -3 \\ 4 & 1 & 2 \end{pmatrix},\ L = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix},$$

$$p = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$ – vector of rows order

$$A = \begin{pmatrix} 2 & 4 & -1 \\ 1 & 1 & -3 \\ \underset{\text{pivot}}{\textcircled{4}} & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 1 & 2 \\ 1 & 1 & -3 \\ 2 & 4 & -1 \end{pmatrix},\ p = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

The first line is multiplied by $-\frac{1}{4}$ and added to the second line.
The first line is multiplied by $-\frac{1}{2}$ and added to the third line.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & * & 1 \end{pmatrix},\ A \to \begin{pmatrix} 4 & 1 & 2 \\ 0 & \frac{3}{4} & -\frac{7}{2} \\ 0 & \underset{\text{pivot}}{\textcircled{\frac{7}{2}}} & -2 \end{pmatrix} \to \begin{pmatrix} 4 & 1 & 2 \\ 0 & \frac{7}{2} & -2 \\ 0 & \frac{3}{4} & -\frac{7}{2} \end{pmatrix}$$

$$L \to \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & * & 1 \end{pmatrix},\ p \to \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

The second line is multiplied by $-\frac{3}{14}$ and added to the third line.

$$A \to \begin{pmatrix} 4 & 1 & 2 \\ 0 & \frac{7}{2} & -2 \\ 0 & 0 & -\frac{43}{14} \end{pmatrix} = U,\ L \to \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{3}{14} & 1 \end{pmatrix}.$$

Theorem

If all main minors of the matrix A are non-zero then it is possible to do Gaussian elimination method without row exchange and the LU decomposition has form

$$A = L \cdot U$$