Numerical methods – lecture 7

Jiří Zelinka

Autumn 2017

Interpolation

$x_0, \ldots, x_n$ – given points, $x_i \neq x_j$ for $i \neq j$
$f_0, \ldots, f_n$ – given function values (measurments), $f_i = f(x_i)$
$\Phi(x) = a_o\Phi_0(x) + \cdots + a_n\Phi_n(x)$ – given function depending on the parameters $a_0, \ldots, a_n$.

Examples:

$\Phi(x) = a_0 + a_1 x + \cdots + a_n x^n$: a polynomial,
$\Phi(x) = a_0 + a_1 e^{ix} + \cdots + a_n e^{inx}$: a trigonometric polynomial.

Problem of interpolation:
find the parameters $a_0, \ldots, a_n$ to fulfill conditions

$$\Phi(x_i) = f_i, \text{ for } i = 0, 1, \ldots, n.$$

Theorem

For given points $(x_i, f_i), i = 0, \dots, n,\ x_i \neq x_j$ for $i \neq j$ there exists the unique polynomial P of degree at most n with

$$P(x_i) = f_i, \quad i = 0, \dots, n.$$

Uniqueness:

If $P_1(x_i) = P_2(x_i) = f_i, \quad i = 0, \dots, n.$, then $Q = P_1 - P_2$ is a polynomial of degree at most n and $Q(x_i) = 0, \quad i = 0, \dots, n.$, i.e., Q has $n+1$ roots so Q must be zero polynomial.

Existence Construction of P:
We construct the polynomials L_i:

- L_i is a polynomial of degree n,
- $L_i(x_j) = \begin{cases} 0 & \text{pro } i \neq j \\ 1 & \text{pro } i = j. \end{cases}$

Points $x_j, j \neq i$ are roots of L_i:

$$L_i(x) = A_i(x - x_0) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n).$$

or

$$L_i(x) = A_i \pi_i(x), \text{ where } \pi_i(x) = \prod_{j \neq i} (x - x_j)$$

$$L_i(x_i) = 1 \ \Rightarrow \ A_i = \frac{1}{\pi_i(x_i)}.$$

$$L_i(x) = \frac{\pi_i(x)}{\pi_i(x_i)} = \frac{\prod\limits_{j\neq i}(x - x_j)}{\prod\limits_{j\neq i}(x_i - x_j)}$$

L_i – Lagrange base polynomials

Lagrange interpolation polynomial:

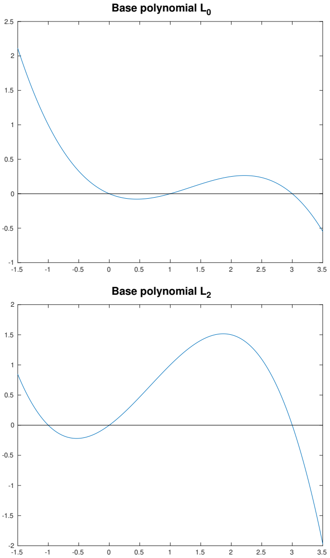
$$P(x) = \sum_{i=0}^{n} f_i L_i(x) = \sum_{i=0}^{n} f_i \frac{\prod\limits_{j\neq i}(x - x_j)}{\prod\limits_{j\neq i}(x_i - x_j)}$$

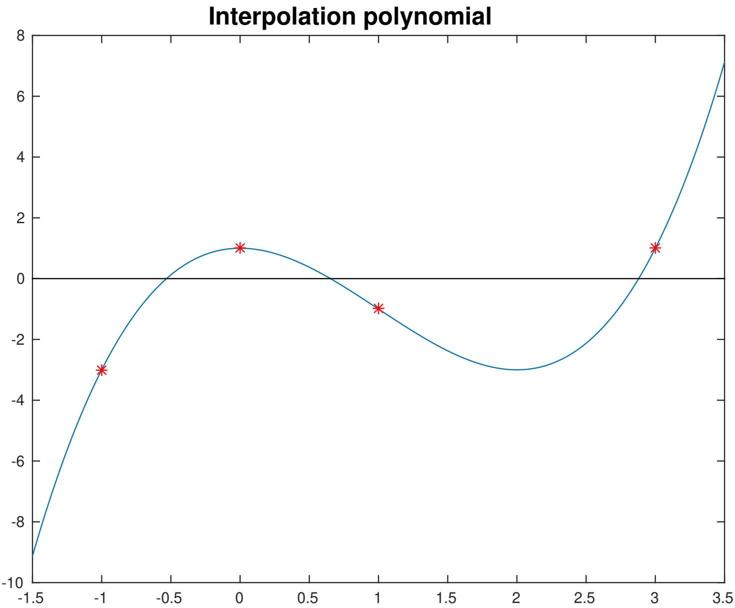
Example:

x_i	-1	0	1	3
f_i	-3	1	-1	1

$$
\begin{aligned}
L_0(x) &= \frac{(x-0)(x-1)(x-3)}{(-1-0)(-1-1)(-1-3)} = -\frac{1}{8}x^3 + \frac{1}{2}x^2 - \frac{3}{8}x \\
L_1(x) &= \frac{(x+1)(x-1)(x-3)}{(0+1)(0-1)(0-3)} = \frac{1}{3}x^3 - x^2 - \frac{1}{3}x + 1 \\
L_2(x) &= \frac{(x+1)(x-0)(x-3)}{(1+1)(1-0)(1-3)} = -\frac{1}{4}x^3 + \frac{1}{2}x^2 + \frac{3}{4}x \\
L_3(x) &= \frac{(x+1)(x-0)(x-1)}{(3+1)(3-0)(3-1)} = \frac{1}{24}x^3 - \frac{1}{24}x \\
P(x) &= -3L_0(x) + L_1(x) - L_2(x) + L_3(x) = x^3 - 3x^2 + 1
\end{aligned}
$$

Lagrange base polynomials




Interpolation polynomial
8
6
4
2
0
-2
-4
-6
-8
-10
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5

Effective calculation of L_i

Calculation of one base polynomial L_i is $O(n^2)$, i.e. direct calculation of the interpolation polynomial is $O(n^3)$.

Effective calculation:

$\omega(x) = \prod_{j=0}^{n} (x - x_j)$	$O(n^2)$
$\pi_i(x) = \omega(x) : (x - x_i)$	Horner scheme, $O(n)$
$pi(x_i)$	Horner scheme, $O(n)$
P	$O(n^2)$

Example:

x_i	-1	0	1	3

$$\begin{aligned}
\omega(x) &= (x+1)(x-0)(x-1)(x-3) = x^4 - 3x^3 - x^2 + 3x \\
\pi_0(x) &= \omega(x) : (x+1) \\
\pi_1(x) &= \omega(x) : (x+0) \\
\pi_2(x) &= \omega(x) : (x-1) \\
\pi_3(x) &= \omega(x) : (x-3)
\end{aligned}$$

Horner scheme for division $\omega(x) : (x - x_0)$, i.e. $\omega(x) : (x+1)$:

ω	1	-3	-1	3	0
-1	$\boxed{1}$	$\boxed{-4}$	$\boxed{3}$	$\boxed{0}$	0

$\pi_0(x) = x^3 - 4x^2 + 3x$

Horner scheme for $\pi_0(x_0) = \pi_0(-1)$:

$\pi(x)$	1	-4	3	0
-1	1	-5	8	$\boxed{-8}$

$\pi_0(-1) = -8$

$L_0(x) = \frac{\pi_0(x)}{\pi_0(x_0)} = -\frac{1}{8}(x^3 - 4x^2 + 3x)$

Similarly $L_1, L_2, \ldots$

Disadvantage of the Lagrange interpolation polynomial: adding a point (x_{n+1}, f_{n+1}) will cause recalculation of all base polynomials L_i.

Newton interpolation polynomial

Base functions:

$\Phi_0(x) = 1,$
$\Phi_1(x) = (x - x_0),$
$\Phi_2(x) = (x - x_0)(x - x_1),$
$\vdots$
$\Phi_n(x) = (x - x_0) \cdots (x - x_{n-1}).$

Interpolation polynomial:

$$P_n(x) = a_o\Phi_0(x) + \cdots + a_n\Phi_n(x)$$

Adding a point (x_{n+1}, f_{n+1}):

$$P_{n+1}(x) = P_n(x) + a_{n+1}\Phi_{n+1}(x)$$

Calculation of parameters a_i:

$a_i = f[x_0, x_1, \ldots, x_i]$ – *divided diffrerence*

$f[x_i] = f_i$

$f[x_i, x_j] = \frac{f_i - f_j}{x_i - x_j}$

$f[x_j, \ldots, x_{j+k}] = \frac{f[x_{j+1}, \ldots, x_{j+k}] - f[x_j, \ldots, x_{j+k-1}]}{x_{j+k} - x_j}$

i.e.

$f[x_0, \ldots, x_i] = \frac{f[x_1, \ldots, x_i] - f[x_0, \ldots, x_{i-1}]}{x_i - x_0}$

$$
\begin{aligned}
P(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \\
&+ \cdots + f[x_0, \ldots, x_n](x - x_0) \cdots (x - x_{n-1})
\end{aligned}
$$

Table of divided diffrerences

x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$\dots$	
x_0	f_0				
		$f[x_0, x_1]$			
x_1	f_1		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$			
x_2	f_2		$\vdots$	$\cdots$	$f[x_0, \ldots, x_n]$
		$\vdots$			
$\vdots$	$\vdots$		$f[x_{n-2}, x_{n-1}, x_n]$		
		$f[x_{n-1}, x_n]$			
x_n	f_n				

Example:

$$
\begin{array}{llccc}
x_i & f_i & f[x_i, x_{i+1}] & f[x_i, x_{i+1}, x_{i+2}] & f[x_0, x_1, x_2, x_3] \\
\hline
-1 & \boxed{-3} & & & \\
 & & > \frac{1+3}{0+1} = \boxed{4} & & \\
0 & 1 & & > \frac{-2-4}{1+1} = \boxed{-3} & \\
 & & > \frac{-1-1}{1-0} = -2 & & > \frac{1+3}{3+1} = \boxed{1} \\
1 & -1 & & > \frac{1+2}{3-0} = 1 & \\
 & & > \frac{1+1}{3-1} = 1 & & \\
3 & 1 & & &
\end{array}
$$

$$
\begin{aligned}
P(x) &= -3 + 4(x+1) - 3(x+1)x + 1(x+1)x(x-1) = \\
&= x^3 - 3x^2 + 1
\end{aligned}
$$