Numerical methods – lecture 8

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Autumn 2017

Spline interpolation

$x_0, \ldots, x_n$ – given points, $x_0 < x_1 < \cdots < x_n$
$f_0, \ldots, f_n$ – given function values

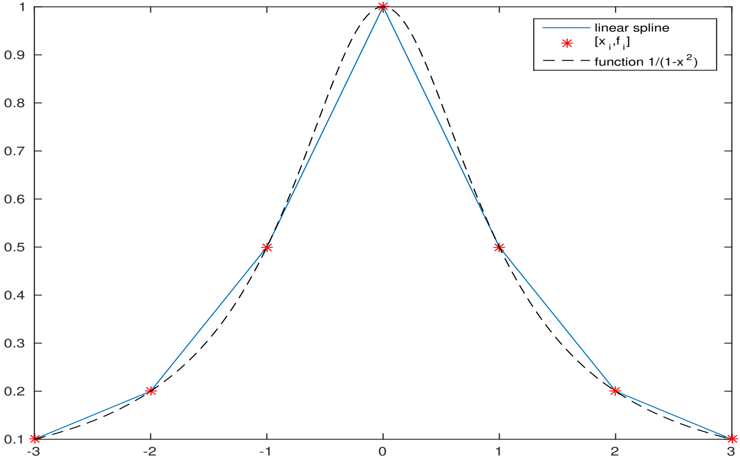
$r, d > 0$ natural numbers, r – degree, d – defect

S – *spline* – piecewise polynomials of degree r
S has continuous derivatives up to order $r - d$

$\mathcal{S}_{r,d}$ – space of splines of degree r with defect d

Example 1.

$\mathcal{S}_{1,1}$ – piecewise linear continuous functions



$S \in \mathcal{S}_{1,1}$ – linear spline, it is determined uniquely by the function values $f_0, \ldots, f_n$.

Example 2.

$\mathcal{S}_{3,1}$ – piecewise polynomials of degree 3 with continuous derivatives to order 2.

Number of parameters describing spline $S \in \mathcal{S}_{3,1}$:

We have n subintervals $I_k = [x_k, x_{k+1}],\ k = 0, \dots, n-1$, in every subinterval the spline is described by 4 parameters:

$$\text{For } x \in I_k \quad S(x) = S_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3$$

$\Rightarrow$ The spline S is described by 4n parameters.

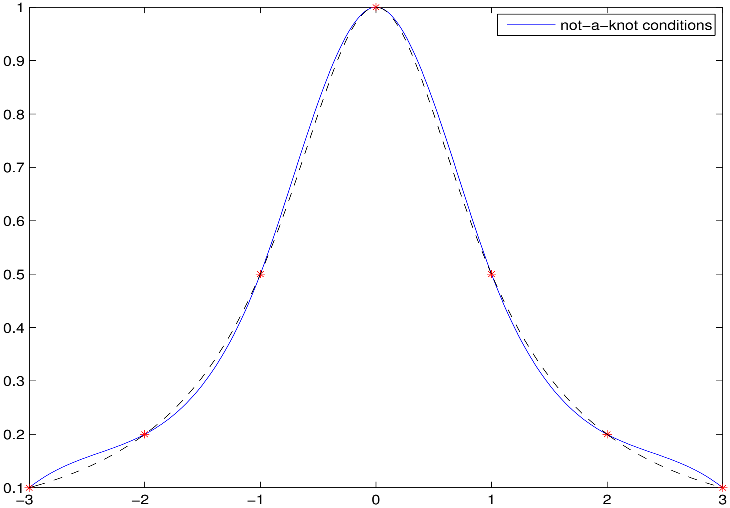
These parameters are bound by conditions:

S is continuous in $x_1, \dots, x_{n-1}$:	$n-1$ conditions
S' is continuous in $x_1, \dots, x_{n-1}$:	$n-1$ conditions
S'' is continuous in $x_1, \dots, x_{n-1}$:	$n-1$ conditions
$S(x_k) = f_k,\ k = 0, \dots, n$:	$n+1$ conditions
Together	$4n-2$ conditions

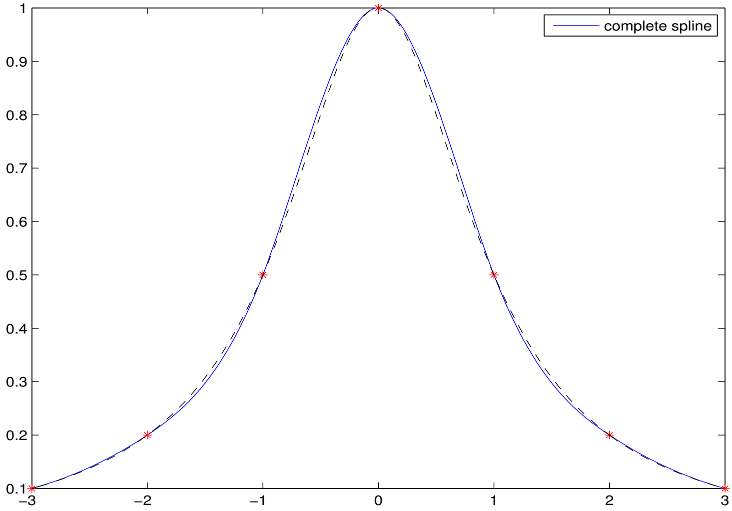
To obtain the unique cubic spline we need two additional *boundary conditions*:

1. $S'(x_0)$ and $S'(x_n)$ are given – complete cubic spline
2. $S'(x_0)$ and $S'(x_n)$ are given, especially $S'(x_0) = S'(x_n) = 0$: natural cubic spline
3. S''' is continuous in x_1 and x_{n-1}: not–a–knot conditions
4. $S(x_0) = S(x_n)$, $S'(x_0) = S'(x_n)$, $S''(x_0) = S''(x_n)$:periodic spline

Interpolation spline with not–a–knot conditions



Interpolation complete spline



Approximation of functions

Bernstein polynomials

$n \in \mathbb{N},\ b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ \ x \in [0,1],\ \ k = 0, \dots, n$

$$\sum_{k=0}^{n} b_{n,k}(x) = 1$$

f is continuous on $[0,1]$, $f_k = f(\frac{k}{n}$

Bernstein polynomial of degree n for the function f:

$$B_{n,f}(x) = \sum_{k=0}^{n} f_k b_{n,k}(x)$$

Theorem

$B_{n,f}$ converges uniformly on $[0,1]$ to the function f for $n \to \infty$.

Theoretical background

$A \cdot x = b$: system of linear equations

For given x let $r_x = b - A \cdot x$: residue for the vector x

$\hat{x}$ is called the *solution in sense of least squares* if
$\|r_{\hat{x}}\| \leq \|r_x\|$ for any x.

$\mathcal{R}(A)$: the range space of the matrix A
$\mathcal{R}^{\perp}(A)$: the orthogonal complement of $\mathcal{R}(A)$
The vector b can be decomposed in the form $b = b_1 + b_2$,
$b_1 \in \mathcal{R}(A)$, $b_2 \in \mathcal{R}^{\perp}(A)$
$A^T \cdot b_2 = o$, o is the zero vector

$\hat{x}$ is the solution of the system

$$A \cdot x = b_1$$

We have

$$A \cdot \hat{x} = b_1$$

$$A^T \cdot A \cdot \hat{x} = A^T \cdot b_1 + o = A^T \cdot b_1 + A^T \cdot b_2 = A^T \cdot b$$

So $\hat{x}$ is the solution of the *system of normal equations*:

$$A^T \cdot A \cdot x = A^T \cdot b$$

Application for the function approximation

$x_0, \ldots, x_n$ – given points
$f_0, \ldots, f_n$ – given function values

$\Phi(x) = c_0\Phi_0(x) + \cdots + c_n\Phi_n(x)$ – given function depending on the parameters $c_0, \ldots, c_n$.

We want to find the parameters $c_0, \ldots, c_n$ to minimize

$$\sum_{k=0}^{n} [\Phi(x_k) - f_k]^2$$

We are looking for the solution in the sense of least squares of the system:

$$\begin{array}{ccccccccc}
c_0\Phi_0(x_0) & + & c_1\Phi_1(x_0) & + & \cdots & + & c_m\Phi_m(x_0) & = & f_0 \\
c_0\Phi_0(x_1) & + & c_1\Phi_1(x_1) & + & \cdots & + & c_m\Phi_m(x_1) & = & f_1 \\
c_0\Phi_0(x_2) & + & c_1\Phi_1(x_2) & + & \cdots & + & c_m\Phi_m(x_2) & = & f_2 \\
 & & & & & & & \vdots & \\
c_0\Phi_0(x_n) & + & c_1\Phi_1(x_n) & + & \cdots & + & c_m\Phi_m(x_n) & = & f_n
\end{array}$$

Let

$$A = \begin{pmatrix} \Phi_0(x_0) & \Phi_1(x_0) & \cdots & \Phi_m(x_0) \\ \Phi_0(x_1) & \Phi_1(x_1) & \cdots & \Phi_m(x_1) \\ \Phi_0(x_2) & \Phi_1(x_2) & \cdots & \Phi_m(x_2) \\ \vdots & & & \vdots \\ \Phi_0(x_n) & \Phi_1(x_n) & \cdots & \Phi_m(x_n) \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

Then the parameters $c = (c_0, \dots, c_m)^T$ are given by the normal equations

$$A^T \cdot A \cdot c = A^T \cdot f$$

i.e.

$$\hat{c} = \left(A^T \cdot A\right)^{-1} A^T \cdot f$$

Example:

x_i	1	2	3	4	5	6	7	8	9	10
f_i	2.7	5.5	7.5	9.0	11.3	12.6	14.9	17.4	19.3	21.5

Find a linear function approximating data.

Solution: $\Phi_0(x) = 1$, $\Phi_1(x) = x$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \\ 1 & 9 \\ 1 & 10 \end{pmatrix}, \qquad f = \begin{pmatrix} 2.7 \\ 5.5 \\ 7.5 \\ 9.0 \\ 11.3 \\ 12.6 \\ 14.9 \\ 17.4 \\ 19.3 \\ 21.5 \end{pmatrix}.$$

$$\hat{c} = \left(A^T \cdot A\right)^{-1} A^T \cdot f \doteq \begin{pmatrix} 1.0267 \\ 2.0261 \end{pmatrix}, \quad \Phi(x) = 1.0267 + 2.0261x\,.$$