Náhodná veličina X má hustotu $f(x) = \begin{cases} \cos(x) & x \in (0, \frac{\pi}{2}) \\ 0 & \text{jinak} \end{cases}$

Určete hustotu náhodné veličiny $Y = X^2$ a $E(Y)$, $D(Y)$.

Hustota Y:

$$g(x) = x^2 \Rightarrow g^{-1}(y) = \sqrt{y}, \quad \frac{dg^{-1}(y)}{dy} = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{dg^{-1}(y)}{dy} \right| = \begin{cases} \frac{\cos(\sqrt{y})}{2\sqrt{y}} & y \in (0, \frac{\pi^2}{4}) \\ 0 & \text{jinak} \end{cases}$$

(transformujeme i interval, kde je hustota nenulová !)

Střední hodnota Y:

$$E(Y) = \int_0^{\frac{\pi^2}{4}} y \cdot \frac{\cos(\sqrt{y})}{2\sqrt{y}} dy \left| \begin{array}{ll} \text{substituce} & \\ t = \sqrt{y} & y = 0 \to t = 0 \\ dt = \frac{1}{2\sqrt{y}} dy & y = \frac{\pi^2}{4} \to t = \frac{\pi}{2} \end{array} \right| =$$

$$= \int_0^{\frac{\pi}{2}} t^2 \cos(t) dt \left| \begin{array}{ll} \text{per partes} & \\ u = t^2 & u' = 2t \\ v' = \cos(t) & v = \sin(t) \end{array} \right| = \left[t^2 \cdot \sin(t) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2t \sin(t) dt =$$

$$= \left(\frac{\pi^2}{4} - 0 \right) + 2 \cdot \int_0^{\frac{\pi}{2}} t \cdot (-\sin(t)) dt \left| \begin{array}{ll} \text{opět per partes} & \\ u = t & u' = 1 \\ v' = -\sin(t) & v = \cos(t) \end{array} \right| =$$

$$= \frac{\pi^2}{4} + 2 \left[\underbrace{\left[t \cos(t) \right]_0^{\frac{\pi}{2}}}_{0 - 0} - \int_0^{\frac{\pi}{2}} \cos(t) dt \right] = \frac{\pi^2}{4} - 2 \cdot \left[\sin t \right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4} - 2 = \underline{\underline{\frac{\pi^2 - 8}{4}}}$$

Pro rozptyl je potřeba vypočítat $E(Y^2)$:

$$E(Y^2) = \int_0^{\pi^2/4} y^2 \frac{\cos(\sqrt{y})}{2\sqrt{y}} dy \left| \begin{array}{ll} \text{substituce} & \\ t = \sqrt{y} & y = 0 \to t = 0 \\ dt = \frac{1}{2\sqrt{y}} dy & y = \frac{\pi^2}{4} \to t = \frac{\pi}{2} \end{array} \right| =$$

$$= \int_0^{\frac{\pi}{2}} t^4 \cos(t) dt \left| \begin{array}{ll} u = t^4 & u' = 4t^3 \\ v' = \cos(t) & v = \sin(t) \end{array} \right| = \left[t^4 \sin(t) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 4t^3 \sin(t) dt =$$

$$= \frac{\pi^4}{16} + 4 \int_0^{\frac{\pi}{2}} t^3 \cdot (-\sin(t)) dt \left| \begin{array}{ll} u = t^3 & u' = 3t^2 \\ v' = -\sin(t) & v = \cos(t) \end{array} \right| = \frac{\pi^4}{16} + 4 \left[\underbrace{\left[t^3 \cos(t) \right]_0^{\frac{\pi}{2}}}_{0 - 0} - \int_0^{\frac{\pi}{2}} 3t^2 \cos(t) dt \right] =$$

$$= \frac{\pi^4}{16} - 12 \underbrace{\int_0^{\frac{\pi}{2}} t^2 \cos(t) dt}_{\text{tento integrál jsme vypočítali u střední hodnoty}} = \frac{\pi^4}{16} - \cancel{12}^{3} \cdot \left(\frac{\pi^2 - 8}{\cancel{4}} \right) = \frac{\pi^4}{16} - 3\pi^2 + 24$$

Dosadíme do vzorce pro rozptyl $D(Y) = E(Y^2) - (E(Y))^2$

$$D(Y) = \frac{\pi^4}{16} - 3\pi^2 + 24 - \left(\frac{\pi^2 - 8}{4}\right)^2 = \frac{\pi^4}{16} - 3\pi^2 + 24 - \frac{\pi^4 - 16\pi^2 + 64}{16} =$$

$$= \frac{\pi^4}{16} - 3\pi^2 + 24 - \frac{\pi^4}{16} + \pi^2 - 4 = \underline{\underline{20 - 2\pi^2}}$$