IA008: Computational Logic

7. Many-Valued Logics

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Basic Concepts

Many-valued logics: Motivation

To some sentences we cannot – or do not want to – assign a truth value since

- they make **presuppositions** that are not fulfilled
  John regrets beating his wife.
  John does not regret beating his wife.
- they refer to **non-existing** objects
  The king of Paris has a pet lion.
- they are too **vague**
  The next supermarket is far away.
- we have **insufficient information**
  The favourite colour of Odysseus was blue.
- we cannot determine their truth
  The Goldbach conjecture holds.

This leads to logics with **truth values** other than 'true' and 'false'.

3-valued logic

truth values ‘false’ $\bot$, ‘uncertain’ u, and ‘true’ $\top$.

A	$\neg A$
$\bot$	$\top$
u	u
$\top$	$\bot$

$\wedge$	$\bot$	u	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
u	$\bot$	u	u
$\top$	$\bot$	u	$\top$

$\vee$	$\bot$	u	$\top$
$\bot$	$\bot$	u	$\top$
u	u	u	$\top$
$\top$	$\top$	$\top$	$\top$

Kleene K3

$\rightarrow$	$\bot$	u	$\top$
$\bot$	$\top$	$\top$	$\top$
u	u	$\boldsymbol{u}$	$\top$
$\top$	$\bot$	u	$\top$

Łukasiewicz L3

$\rightarrow$	$\bot$	u	$\top$
$\bot$	$\top$	$\top$	$\top$
u	u	$\boldsymbol{\top}$	$\top$
$\top$	$\bot$	u	$\top$

Example

A	B	$A \wedge (A \rightarrow B)$	$A \wedge (A \rightarrow B) \rightarrow B$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	u	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$
u	$\bot$	u	u
u	u	u	$u/\top$
u	$\top$	u	$\top$
$\top$	$\bot$	$\bot$	$\top$
$\top$	u	u	$u/\top$
$\top$	$\top$	$\top$	$\top$

Fuzzy logic

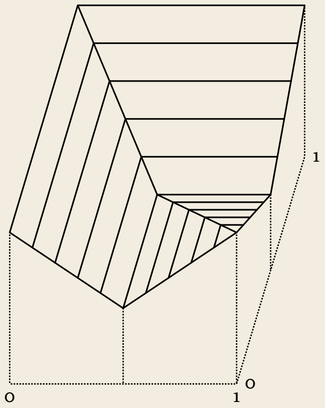
Truth values: $v \in [0, 1]$ measuring **how true** a statement is. 0 means 'false' and 1 means 'true'.

Several possible semantics:

$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$
$1 - A$	$A \cdot B$	$1 - (1 - A)(1 - B)$	$1 - A(1 - B)$
$1 - A$	$\min(A, B)$	$\max(A, B)$	$\max(1 - A, B)$
$1 - A$	$\max(A + B - 1, 0)$	$\min(A + B, 1)$	$\min(1 - A + B, 1)$

Example

$$A \wedge (A \to B) \to B = \max(1 - \min(A, \max(1 - A, B)), B)$$



Tableaux for L3

statements: $t \leq \varphi$, $\varphi \leq t$, $t \nleq \varphi$, or $\varphi \nleq t$, for $t \in \{\bot, u, \top\}$

Construction

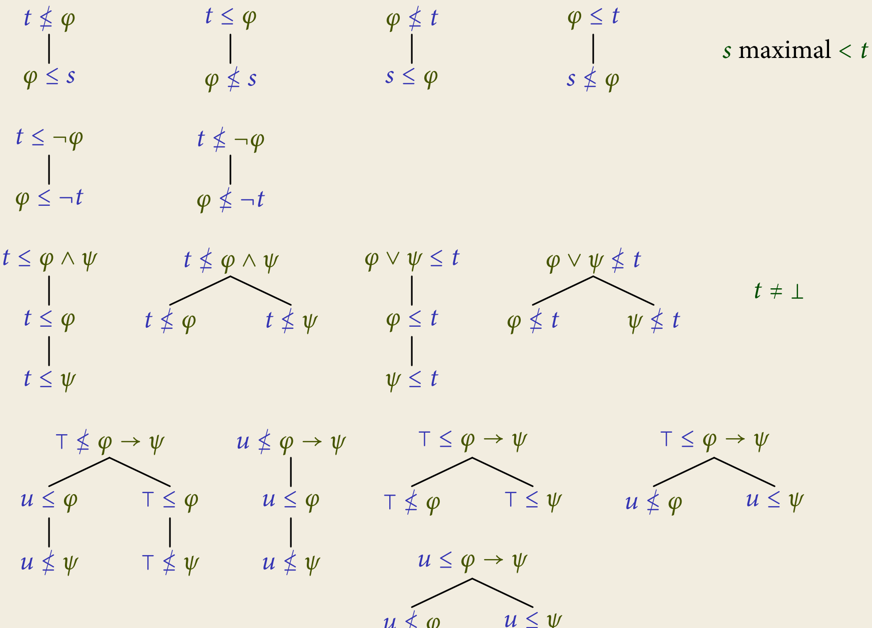
A **tableau** for a formula φ is constructed as follows:

- start with $1 \nleq \varphi$
- choose a branch of the tree
- choose a statement σ on the branch
- choose a rule with head σ
- add it at the bottom of the branch
- repeat until every branch contains one of the following **contradictions**

$$\begin{array}{lll} \bot \nleq \varphi & s \leq t \text{ with } s \nleq t & s \leq \varphi \text{ and } t \nleq \varphi \text{ with } t \leq s \\ \varphi \nleq \top & s \nleq t \text{ with } s \leq t & \varphi \leq s \text{ and } \varphi \nleq t \text{ with } s \leq t \end{array}$$

where $s, t \in \{\bot, u, \top\}$ and φ is a formula

Tableaux Rules



$t \not\leq \varphi$
$\varphi \leq s$

$t \leq \varphi$
$\varphi \not\leq s$

$\varphi \not\leq t$
$s \leq \varphi$

$\varphi \leq t$
$s \not\leq \varphi$

s maximal $< t$

$t \leq \neg\varphi$
$\varphi \leq \neg t$

$t \not\leq \neg\varphi$
$\varphi \not\leq \neg t$

$t \leq \varphi \wedge \psi$
$t \leq \varphi$
$t \leq \psi$

$t \not\leq \varphi \wedge \psi$
$t \not\leq \varphi$ $\quad t \not\leq \psi$

$\varphi \vee \psi \leq t$
$\varphi \leq t$
$\psi \leq t$

$\varphi \vee \psi \not\leq t$
$\varphi \not\leq t$ $\quad \psi \not\leq t$

$t \neq \bot$

$\top \not\leq \varphi \to \psi$
$u \leq \varphi$ $\quad \top \leq \varphi$
$u \not\leq \psi$ $\quad \top \not\leq \psi$

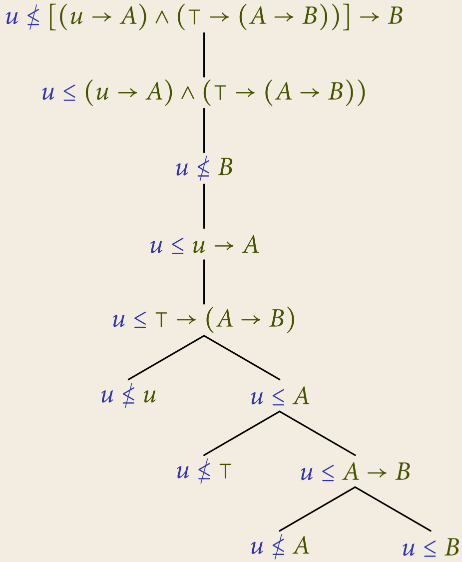
$u \not\leq \varphi \to \psi$
$u \leq \varphi$
$u \not\leq \psi$

$\top \leq \varphi \to \psi$
$\top \not\leq \varphi$ $\quad \top \leq \psi$

$\top \leq \varphi \to \psi$
$u \not\leq \varphi$ $\quad u \leq \psi$

$u \leq \varphi \to \psi$
$u \not\leq \varphi$ $\quad u \leq \psi$

Example



$u \nleq [(u \to A) \wedge (\top \to (A \to B))] \to B$

$u \leq (u \to A) \wedge (\top \to (A \to B))$

$u \nleq B$

$u \leq u \to A$

$u \leq \top \to (A \to B)$

$u \nleq u$

$u \leq A$

$u \nleq \top$

$u \leq A \to B$

$u \nleq A$

$u \leq B$

Intuitionistic Logic

The constructivists view

- We are not interested in **truth** but in **provability.**
- To prove the **existence** of an object is to give a concrete example.

  $$\text{prove } \exists x \varphi(x) \quad \Leftrightarrow \quad \text{find } t \text{ with } \varphi(t)$$

- To prove a **disjunction** is to prove one of the choices.

  $$\text{prove } \varphi \vee \psi \quad \Leftrightarrow \quad \text{prove } \varphi \text{ or prove } \psi$$

Goal

A variant of first-order logic that captures these ideas.

Boolean algebras

In **classical logic** the **truth values** form a **boolean algebra** with operations

$$\wedge,\ \vee,\ \neg,\ \top,\ \bot$$

Properties of negation:

$$x \wedge \neg x = \bot \qquad x \vee \neg x = \top$$

Heyting algebras

In **intuitionistic logic** the **truth values** form instead a **Heyting algebra** with operations

$$\wedge,\ \vee,\ \to,\ \top,\ \bot$$

Properties of implication:

$$z \leq x \to y \quad \text{iff} \quad z \wedge x \leq y$$

(that is $x \to y$ is the largest element satisfying $(x \to y) \wedge x \leq y$)

$$\begin{aligned} x \wedge (x \to y) &= x \wedge y & x \to x &= \top \\ y \wedge (x \to x) &= y & x \to (y \wedge z) &= (x \to y) \wedge (x \to z)\,. \end{aligned}$$

Heyting algebras

In **intuitionistic logic** the **truth values** form instead a **Heyting algebra** with operations

$$\wedge,\ \vee,\ \rightarrow,\ \top,\ \bot$$

Properties of implication:

$$z \leq x \rightarrow y \quad \text{iff} \quad z \wedge x \leq y$$

(that is $x \rightarrow y$ is the largest element satisfying $(x \rightarrow y) \wedge x \leq y$)

$$\begin{aligned} x \wedge (x \rightarrow y) &= x \wedge y & \qquad x \rightarrow x &= \top \\ y \wedge (x \rightarrow x) &= y & \qquad x \rightarrow (y \wedge z) &= (x \rightarrow y) \wedge (x \rightarrow z)\,. \end{aligned}$$

Negation $\neg x := x \rightarrow \bot$

satisfies $x \wedge \neg x = \bot$, but not $x \vee \neg x = \top$

Forcing Frames

Definition

Transition system $\mathfrak{S} = \left\langle S, \leq, (P_i)_{i \in I}, s_0 \right\rangle$ with one edge relation $\leq$ that forms a **partial order:**

- **reflexive** $s \leq s$
- **transitive** $s \leq t \leq u$ implies $s \leq u$
- **anti-symmetric** $s \leq t$ and $t \leq s$ implies $s = t$

The forcing relation

$\mathfrak{S}$ forcing frame, $s \in S$ state, φ formula

$$\begin{array}{lll} s \Vdash P_i & :\text{iff} & t \in P_i \quad \text{for all } t \geq s \\ s \Vdash \varphi \wedge \psi & :\text{iff} & s \Vdash \varphi \text{ and } s \Vdash \psi \\ s \Vdash \varphi \vee \psi & :\text{iff} & s \Vdash \varphi \text{ or } s \Vdash \psi \\ s \Vdash \neg\varphi & :\text{iff} & t \nVdash \varphi \quad \text{for all } t \geq s \\ s \Vdash \varphi \to \psi & :\text{iff} & t \Vdash \varphi \text{ implies } t \Vdash \psi \quad \text{for all } t \geq s \end{array}$$

The **truth value** of φ in $\mathfrak{S}$ is

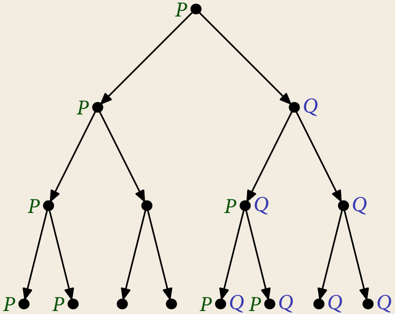
$$[\![\varphi]\!]_{\mathfrak{S}} := \{\, s \in S \mid s \Vdash \varphi \,\}\,,$$

which is **upwards-closed** with respect to $\leq$.

Intuition

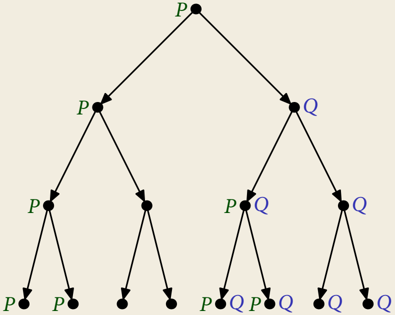
Intuitionistic logic speaks about the **limit behaviour** of φ for large s.

Example



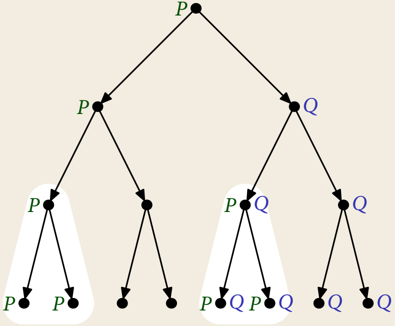
Example

$\varphi := P$



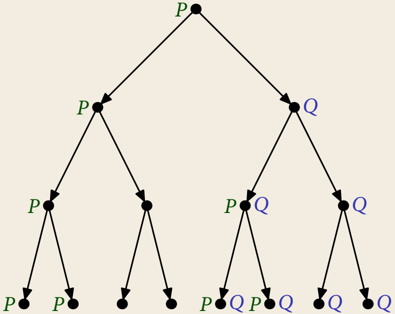
Example

$\varphi := P$



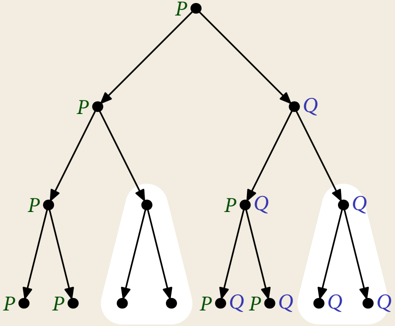
Example

$\varphi := \neg P$



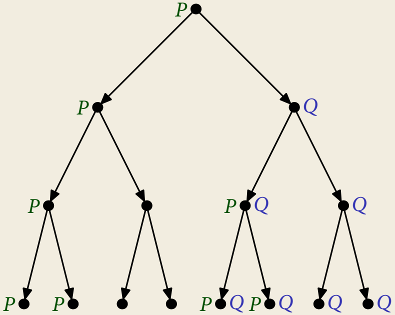
Example

$\varphi := \neg P$



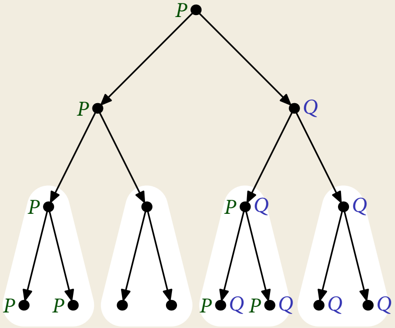
Example

$\varphi := P \vee \neg P$



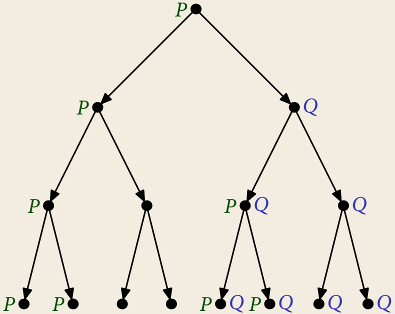
Example

$\varphi := P \vee \neg P$



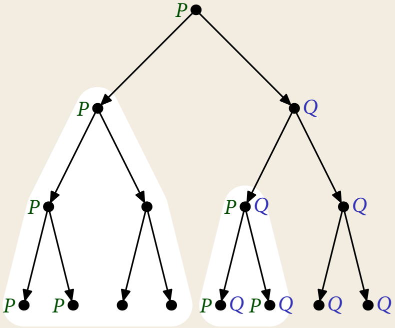
Example

$\varphi := Q \to P$



Example

$\varphi := Q \to P$



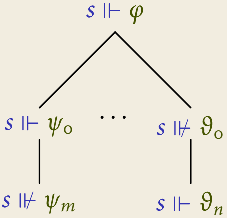
Tableaux for Intuitionistic Logic

Statements

$s \Vdash \varphi \qquad s \nVdash \varphi \qquad s \leq t$

s, t state labels, φ a **formula**

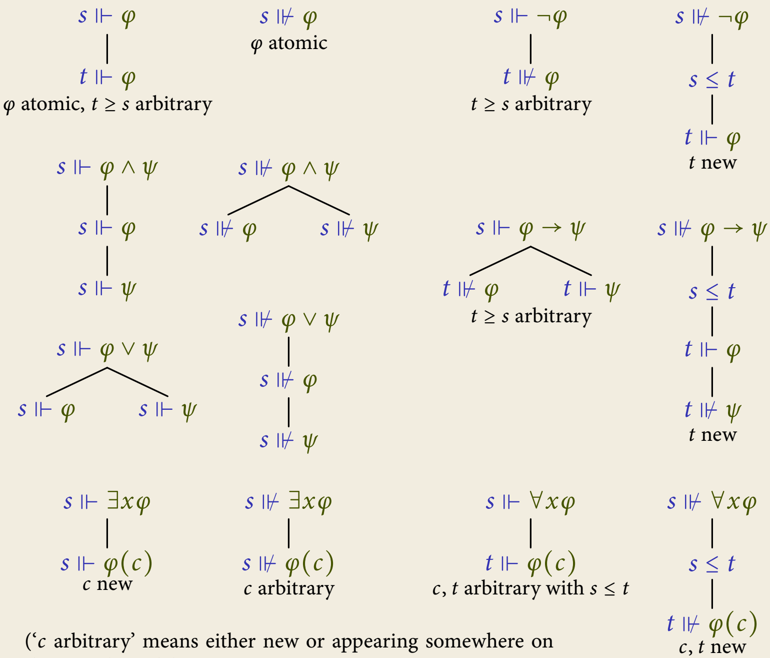
Rules



$s \Vdash \varphi$

$s \Vdash \psi_0 \quad \cdots \quad s \nVdash \vartheta_0$

$s \nVdash \psi_m \qquad s \Vdash \vartheta_n$



(‘c arbitrary’ means either new or appearing somewhere on the same branch.)

$$s \nVdash A \to (B \to A)$$

$$s \leq t$$

$$t \Vdash A$$

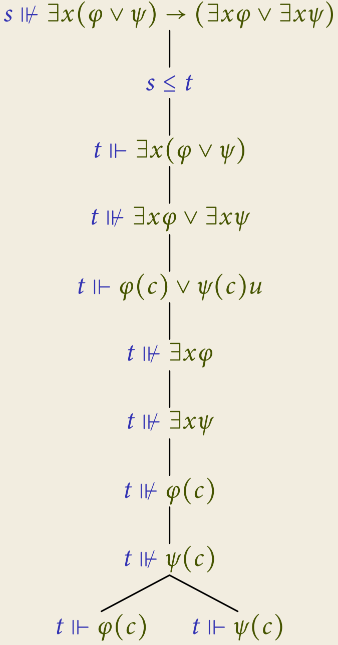
$$t \nVdash B \to A$$

$$t \leq u$$

$$u \Vdash B$$

$$u \nVdash A$$

$$u \Vdash A$$



$s \nVdash \exists x(\varphi \lor \psi) \to (\exists x\varphi \lor \exists x\psi)$

$s \leq t$

$t \Vdash \exists x(\varphi \lor \psi)$

$t \nVdash \exists x\varphi \lor \exists x\psi$

$t \Vdash \varphi(c) \lor \psi(c)u$

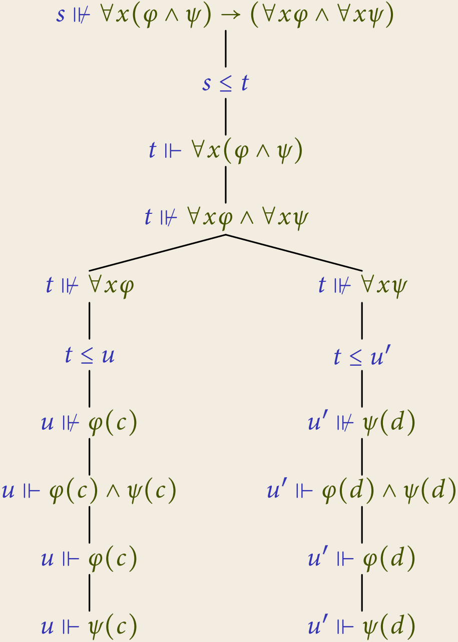
$t \nVdash \exists x\varphi$

$t \nVdash \exists x\psi$

$t \nVdash \varphi(c)$

$t \nVdash \psi(c)$

$t \Vdash \varphi(c)$ $\quad$ $t \Vdash \psi(c)$


$s \nVdash \forall x(\varphi \wedge \psi) \rightarrow (\forall x\varphi \wedge \forall x\psi)$
$s \leq t$
$t \Vdash \forall x(\varphi \wedge \psi)$
$t \nVdash \forall x\varphi \wedge \forall x\psi$
$t \nVdash \forall x\varphi$
$t \nVdash \forall x\psi$
$t \leq u$
$t \leq u'$
$u \nVdash \varphi(c)$
$u' \nVdash \psi(d)$
$u \Vdash \varphi(c) \wedge \psi(c)$
$u' \Vdash \varphi(d) \wedge \psi(d)$
$u \Vdash \varphi(c)$
$u' \Vdash \varphi(d)$
$u \Vdash \psi(c)$
$u' \Vdash \psi(d)$