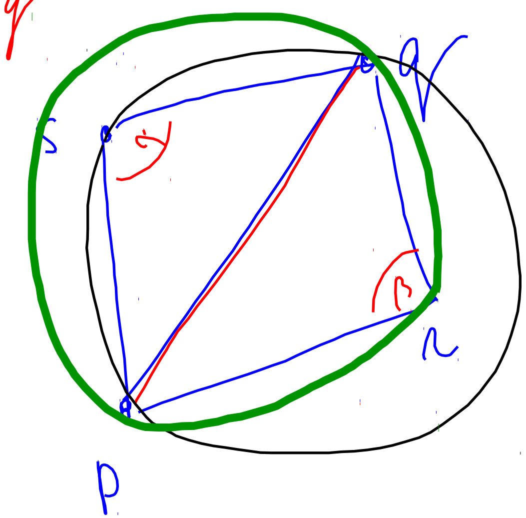
Delaunay triangulation

$P = \{p_1, \dots, p_n\}$

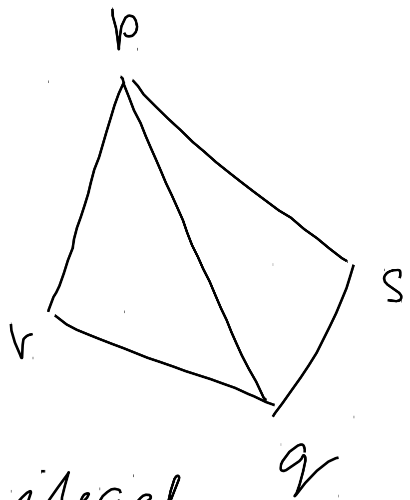
Optimal triangulation .... maximal in lexicographic order

Legal triangulation .... triangulation without illegal edges

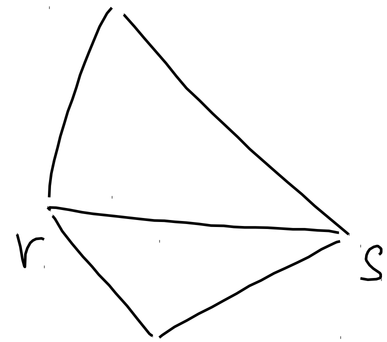
illegal edge pq



$\alpha + \beta > 180°$



flip $\Longrightarrow$



rs is legal

pq illegal

T $\quad$ T' $\quad \Longrightarrow \quad T < T'$

Conclusion was that every optimal triangulation is legal.

Opposite statement is not true. - figure 10.6.

Delaunay triangulation - we introduce this notion in 2 steps

Delaunay graph 2 equivalent definition

The couple $p_i, p_j \in P$ forms an edge of D. graph,

if there is a circle on which p_i, p_j are lying and

and the other points from P are lying outside the circle.

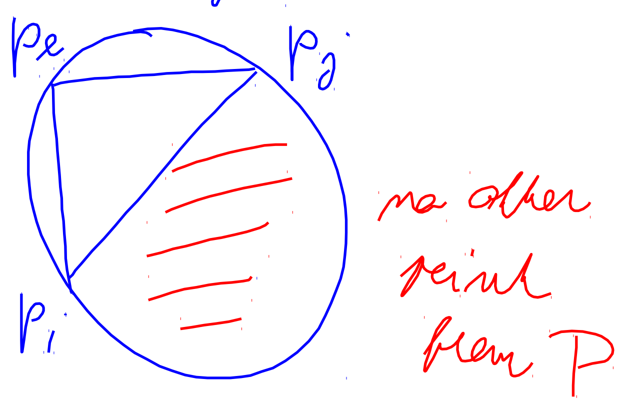
Consider a planar graph G (Voronoi diagram in our case).
Dual graph has faces of G as vertices and edges in G as edges between faces in the new diagram.
From this definition of dual graph follows that Delaunay graph is the dual graph to Voronoi diagram.
This is the second (equivalent) definition of D. graph.
Dual graph to Voronoi diagram is planar graph.
So Delaunay diagram is planar graph.
Delaunay triangulation is an arbitrary triangulation of Delaunay graph.

Direct characterization of D. triangulations is

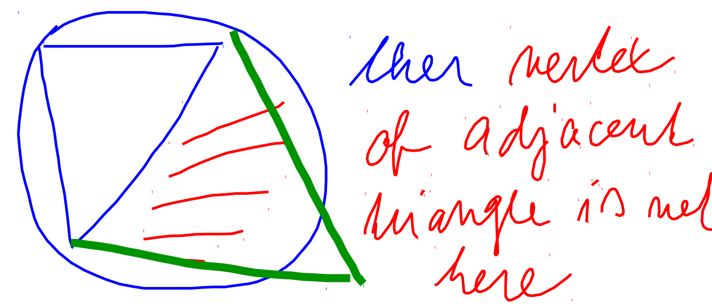
Lemma

Triangulation is Delaunay if and only if every its edge $p_i p_j$ has the property that in the interior of the circle circumscribed to a triangle $p_i p_j p_\ell$ there is no other point from the set P.

Delaunay

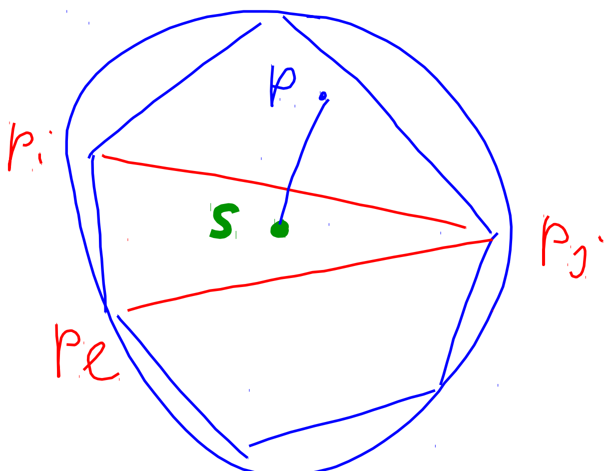


legal



<u>Proof</u> : Let we have D. triangulation

$\Delta p_i p_j p_e$ is a triangle which has arised from the triangulation of a polygon in D. graph



If there is a point from in the interior, then s would not be a vertex of Voronoi diagram

$\Leftarrow$ Let there is ~~x~~ no point in the circle circumscribed to $\Delta p_i p_j p_e$. It means that $p_i p_j p_e$ are vertices of a polygon from the D graph. So the triangle is in the D. triangulation

Theorem The set of D. triangulations is the same as the set of legal triangulations.

Proof: All D. triangulation are legal.
Follows from the characterization of D. triangulation and the definition of legal triangulation.

Every legal triangulation is Delaunay. By contradiction.
Let us have legal triangulation which has an edge $p_i p_j$ and triangle $p_i p_j p_k$ such that there is a point p_e inside the circle circumscribed to $\Delta p_i p_j p_k$.
Let us choose the triangle $\Delta p_i p_j p_k$ and p_e in such a way that the angle $\measuredangle p_i p_e p_j$ is maximal.
The rest according to figure 10.11.

Algorithm for D. triangulation

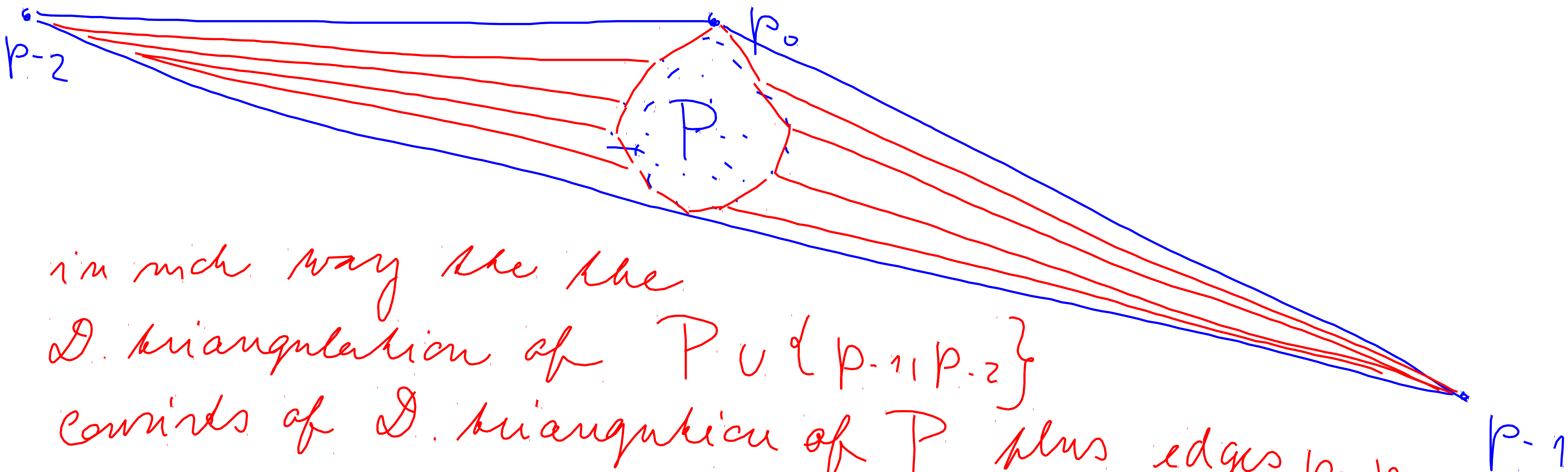
(1) Take an algorithm for Voronoi diagram and make a dual graph and triangulate polygons if they have more than 3 edges.

(2) Naive algorithm .... we take a triangulation and we subsequently remove the illegal edges by flips. This procedure will finish. Time complexity is bad.

(3) Randomized incremental algorithm

Take a set $P = \{p_0, p_1, \dots p_n\}$ such that p_0 has the biggest y-coordinate (biggest x-coordinate).

1st step Add another two points p_{-1}, p_{-2} to the set P

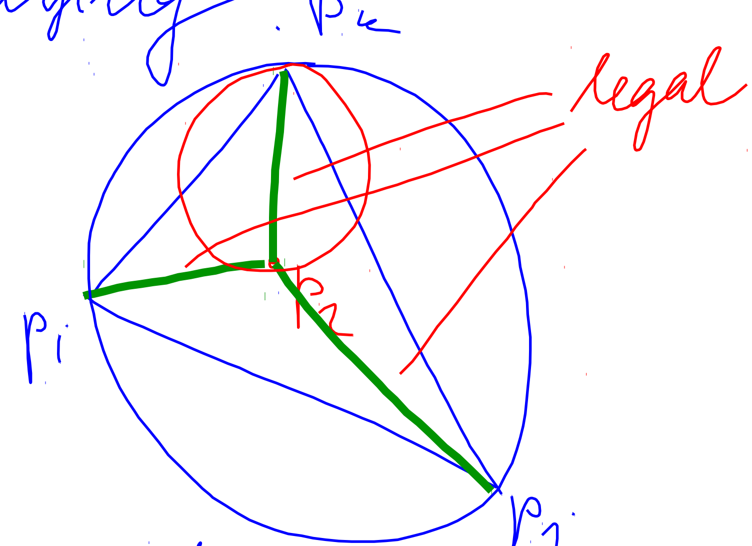


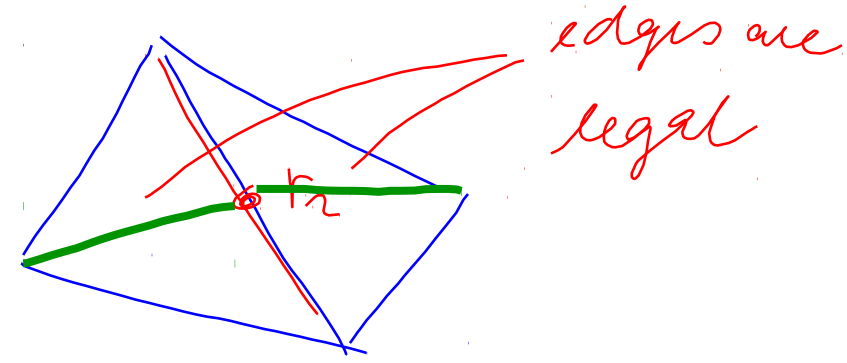
in such way that the
D. triangulation of $P \cup \{p_{-1}, p_{-2}\}$
consists of D. triangulation of P plus edges $p_0 p_{-1}$, $p_0 p_{-2}$,
$p_{-2} p_{-1}$ and edges p_{-2} vertex of left part of the convex hull,
p_{-1} vertex of the right part of the convex hull.

2nd step Let us suppose we have D. triangulation T_{r-1} for the set $P_{r-1} = \{p_{-2}, p_{-1}, p_0, \dots, p_{r-1}\}$ $r \geq 1$.

Moreover, let the order $p_1, p_2, \dots, p_n$ be random.

Using special search structure $\mathcal{D}_{r-1}$ we find a triangle in T_{r-1} or an edge in T_{r-1} where p_r is lying





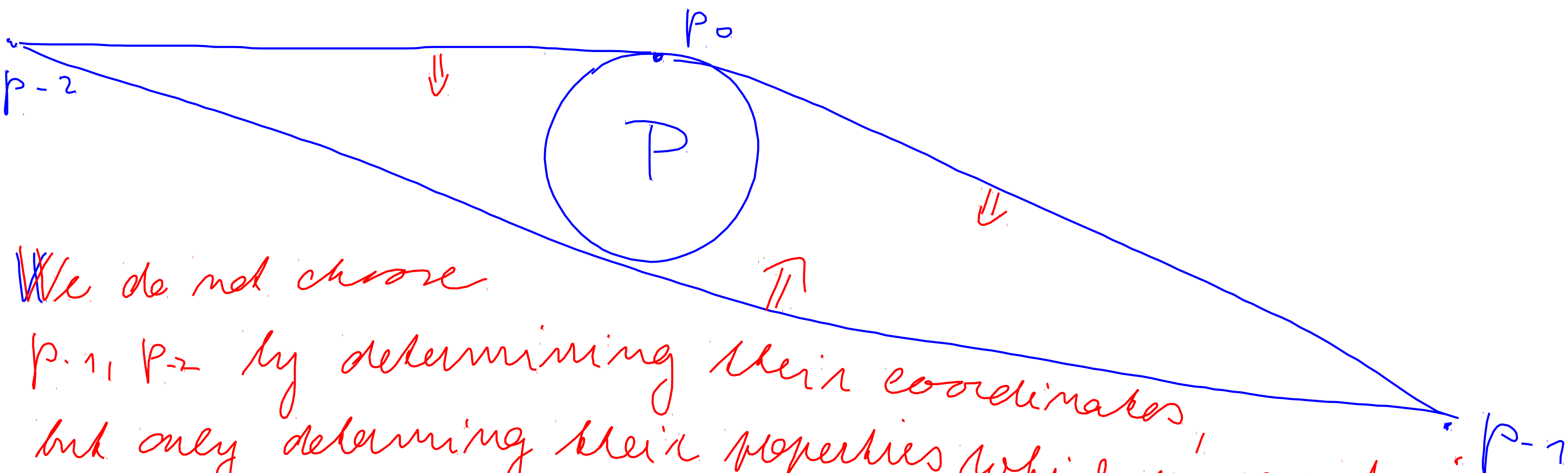
If some of edges $p_i p_j$, $p_i p_k$, $p_j p_k$ is not legal, make a flip.
And proceed in this as far as there is an illegal edge.

3rd step   Remove points p_{-2}, p_{-1} and edges $p_{-1}p_i, p_{-2}p_j$ from the triangulation.

A search structure is an oriented graph which has triangles of the triangulation as leaves and the triangles of previous triangulations as inner nodes. See figures

Theorem 10.1 Expected time for the algorithm for Pwitting $n+1$ points is $O(n \log n)$.

The choice of points p_{-1}, p_{-2}



We do not choose p_{-1}, p_{-2} by determining their coordinates, but only determining their properties which we need for the algorithm.

1. p_{-1} lies outside all circles circumscribed to points $p_0, \dots, p_n$
2. all points from P lie under the line $p_0 p_{-1}$
3. p_{-2} lies outside all circles circumscribed to points $p_{-1}, p_0, \dots, p_n$
4. all points from P lie over the line $p_{-2} p_{-1}$
5. all points from P lie under the line $p_{-2} p_0$.