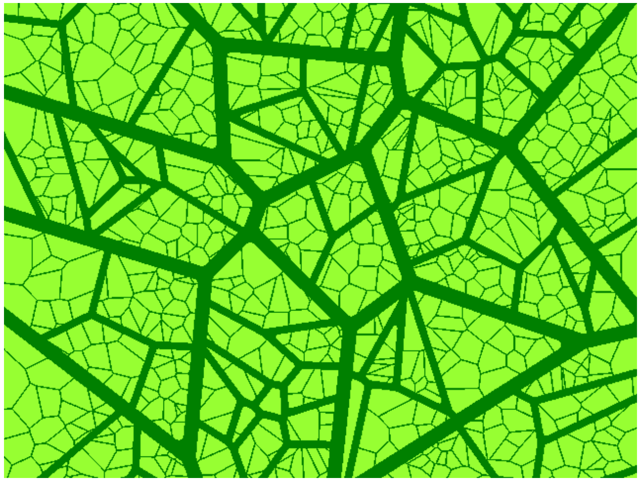
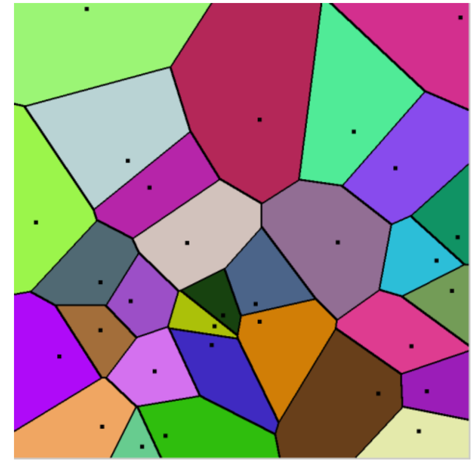
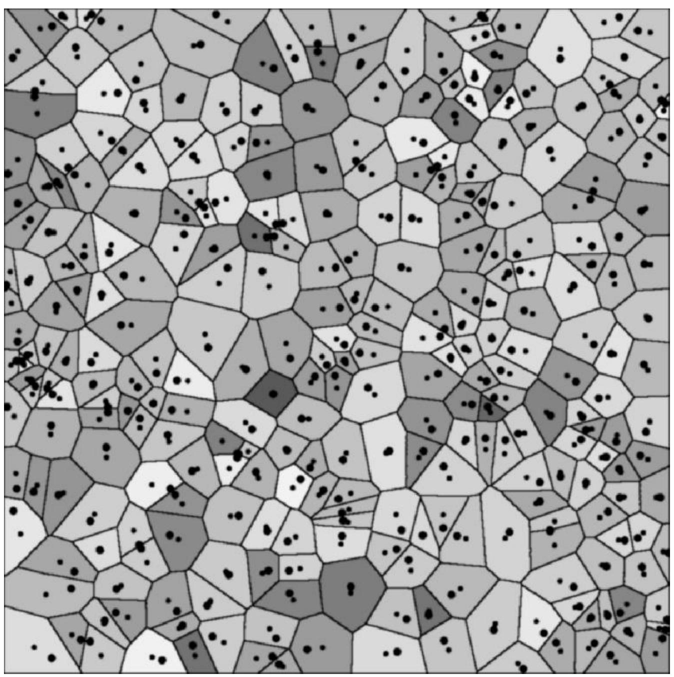


www.grasshopper3d.com

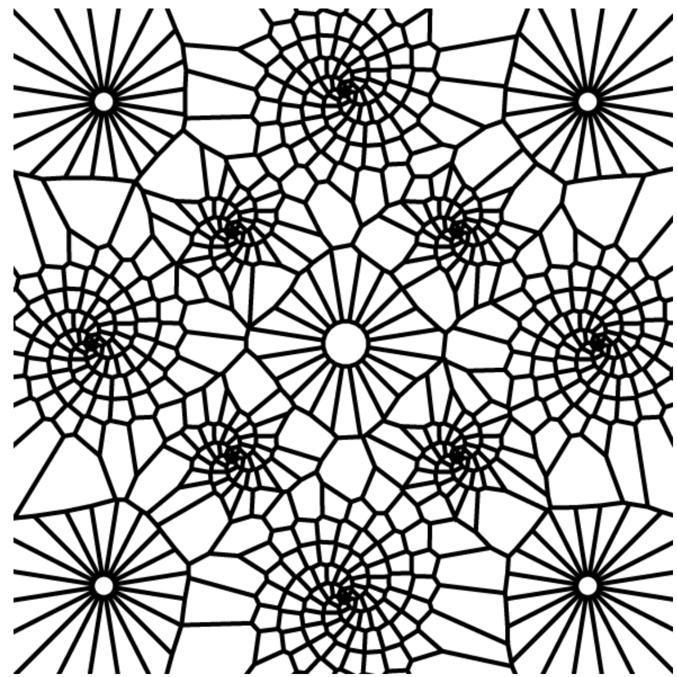


www.sonycsl.co.jp

Voronoi diagrams



cs.nyu.edu



http://newtextiles.media.mit.edu/?p=1906

Motivation

- Solves so-called **post office problem**
  - The goal is to plan a placement of new post office/supermarket/…
  - How many people will find the new supermarket attractive?
  - Lets consider the following simplified requirements:
    - The price of all goods is the same in all supermarkets
    - Total cost = cost for the goods + travelling cost to the supermarket
    - Travelling cost to the supermarket = Euclidean distance to the supermarket x fixed cost per distance unit
    - The goal of the customer is to minimize the costs
  - Consequence: the customers are using the service of the nearest supermarket

Motivation

- This model induces the division of the space to subregions according to the location of the supermarkets – each subregion contains **all points** being **closer** to the given supermarket than to any other supermarket
- Such a space division is called **Voronoi diagram**

Euclidean distance

- **Euclidean distance** between two points $P = [p_x, p_y]$ and $Q = [q_x,q_y]$ is defined as

$$|PQ| = \text{dist}(P,Q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$$

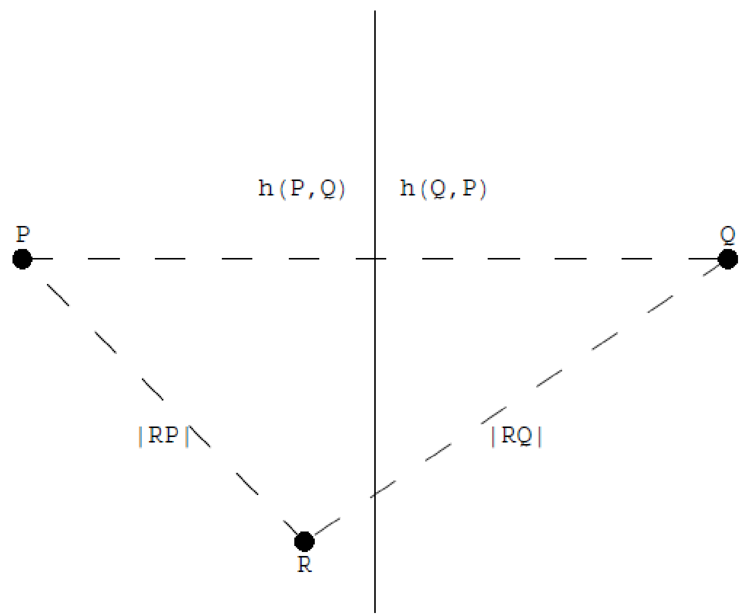
VD definition

- Let $P = \{P_1, \ldots, P_n\}$ be a set of n different points in space, called **generating points**.
- Voronoi diagram of P is the division to n cells connected with points P_i in that way that an arbitrary point Q lies in the cell of P_i only when

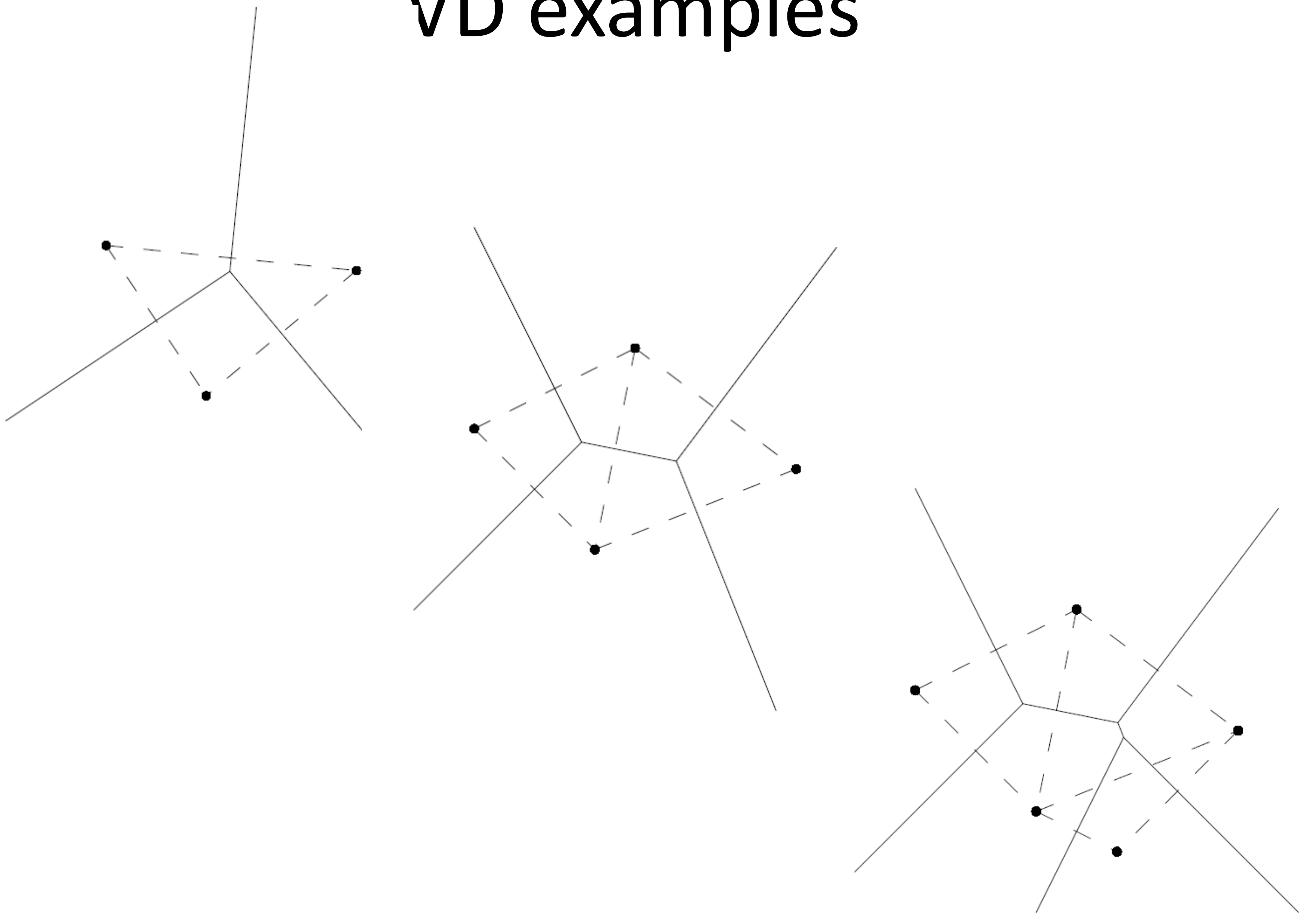
$$|QP_i| < |QP_j| \quad \text{for all } P_j \in P,\ j \neq i$$

VD definition

- Lets denote the Voronoi diagram of *P* as **Vor(*P*)**
- A cell of Vor(*P*), belonging to point *Pi*, is denoted as y(P_i) and we call it a **Voronoi cell of point P_i**

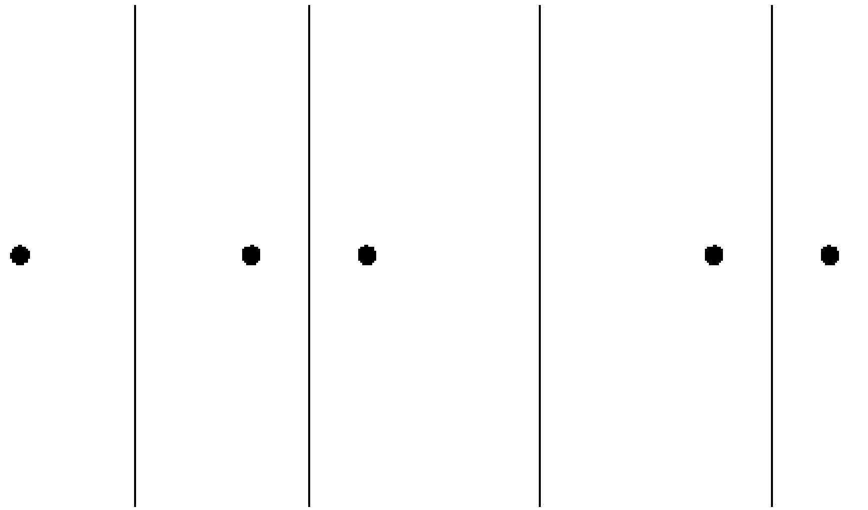


VD examples



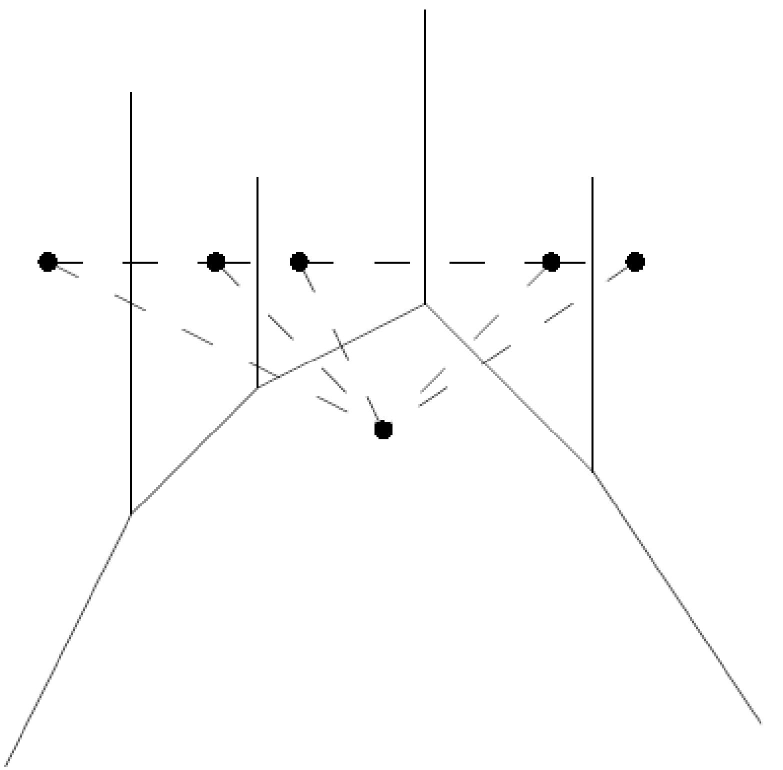
VD properties

- If all points in *P* are colinear, Vor(*P*) consists of $n - 1$ parallel lines



VD properties

- If the points are not colinear, Vor(*P*) is continuous and its edges are line segments or half-segments

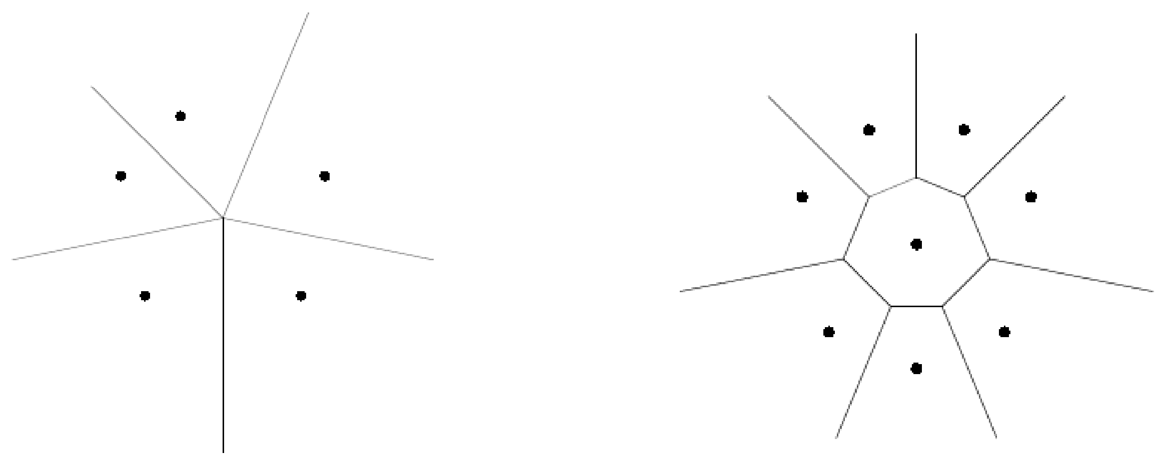


VD properties

- Voronoi cell y(P_i) is **unlimited** only when the point P_i belongs to an edge of the convex hull of P

VD properties

- If *P* contains 4 or more vertices lying on one circle, there is a Voronoi vertex formed by the intersection of Voronoi edges whose number corresponds to the number of points on that circle – we call it a **degenerated Voronoi diagram**



Algorithms for VD construction

- Generally, creating VD for *n* points lies in *O(n log n)*
- Algorithms:
  - Naïve approach
  - Incremental algorithm
  - Divide and conquer
  - Sweep line (Fortune's algorithm)

  …

Naïve approach

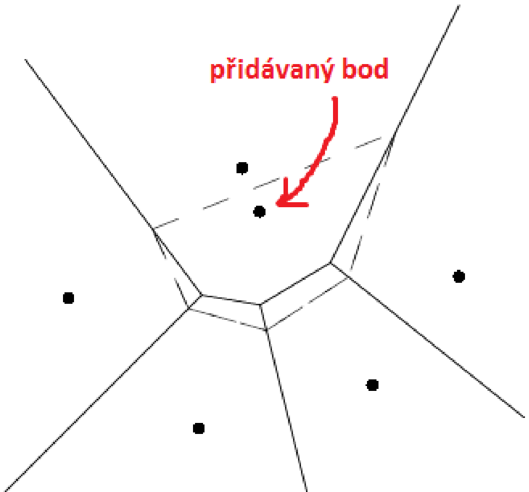
- Each region y(P_i) of Voronoi diagram is generated as an intersection between halfplanes $h(P_i, P_j)$, for all $j \neq i$.
- The complexity of finding one region = $O(n \log n)$
- Total complexity = $\boldsymbol{O(n^2 \log n)}$

Incremental algorithm

1. For all points *P*:
   1. In the current VD, we localize the corresponding Voronoi cell containing P_{i+1} → y(P_{i1})
   2. We create the axis of line segment $P_{i+1}P_{i1}$
   3. We determine the intersections of this axis of line segment $P_{i+1}P_{i1}$ with the boundary of y(P_{i1})
   4. We select one of the intersections which determines the Voronoi cell with which our algorithm will continue in the next step → y(P_{i2})

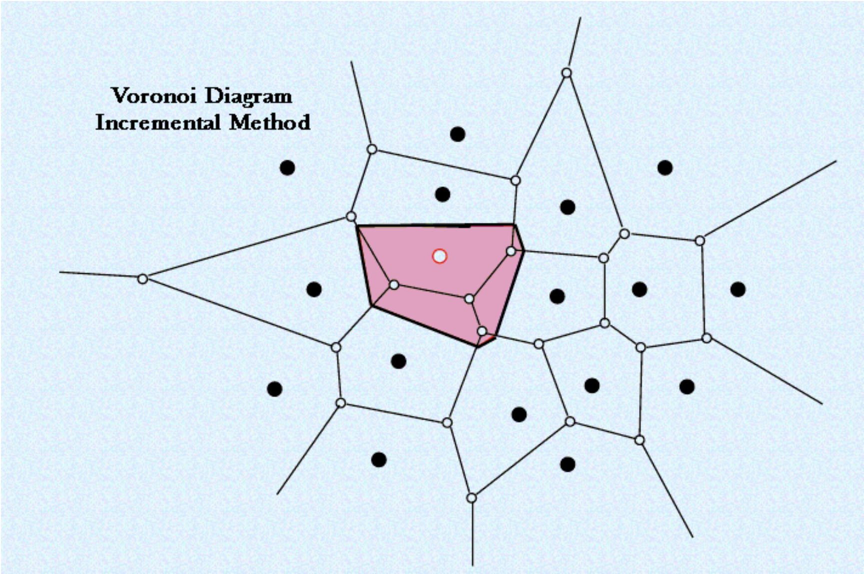
Incremental algorithm

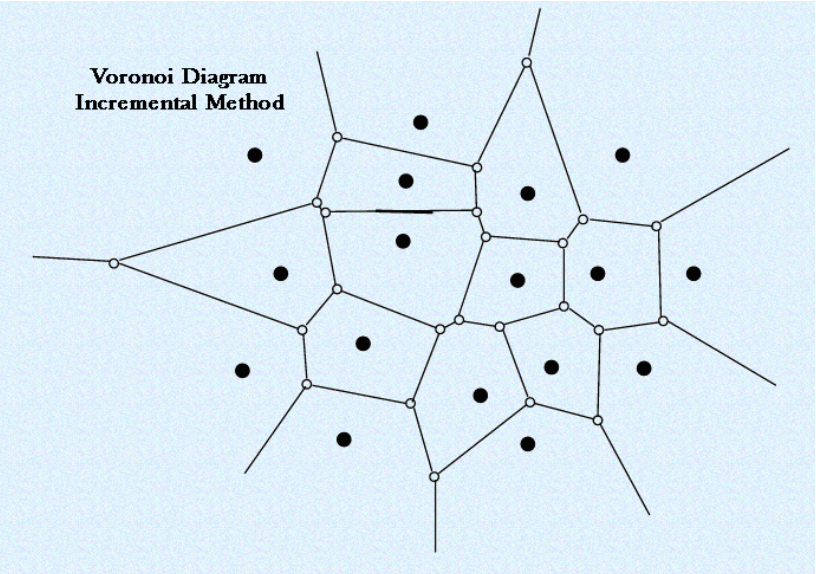
5. We create the axis of the line segment $P_{i+1}P_{i2}$ and its intersections with the boundary of y(P_{i2}). We select an intersection not lying on the common edge of y(P_{i1}) and y(P_{i2}) and we continue
6. We repeat step 5, until we reach the second intersection of the axis of line intersection $P_{i+1}P_{i1}$ with the boundary of y(P_{i1})
7. We remove the edges inside the newly created Voronoi cell



Incremental algorithm

- Complexity $O(n^2)$, in special cases even $O(n)$





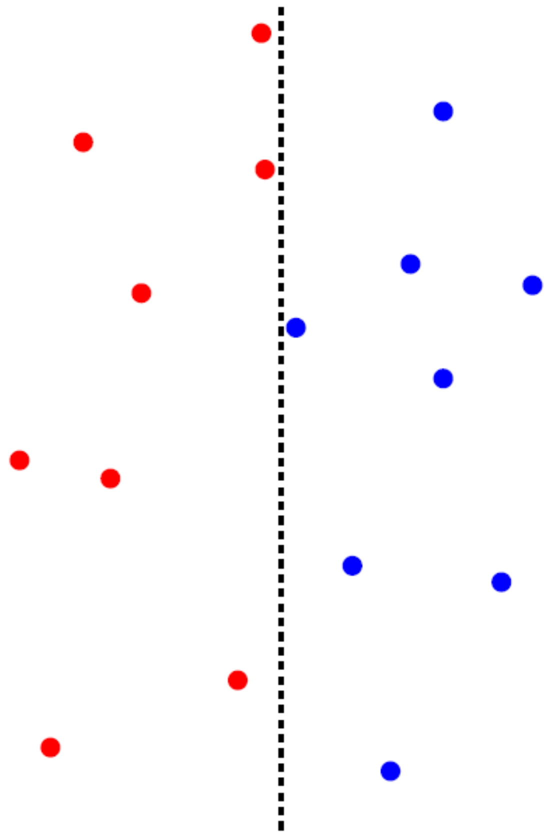
http://www.personal.kent.edu/~rmuhamma/Compgeometry/MyCG/Voronoi/Incremental2/incremental2.htm

Divide and conquer

- The input set is recursively divide to two subsets until we reach the set of three points for which we construct the VD easily
- The crucial part is the „backtracking step“, where the individual solutions have to be merged to one VD
- Complexity *O(n log n)*

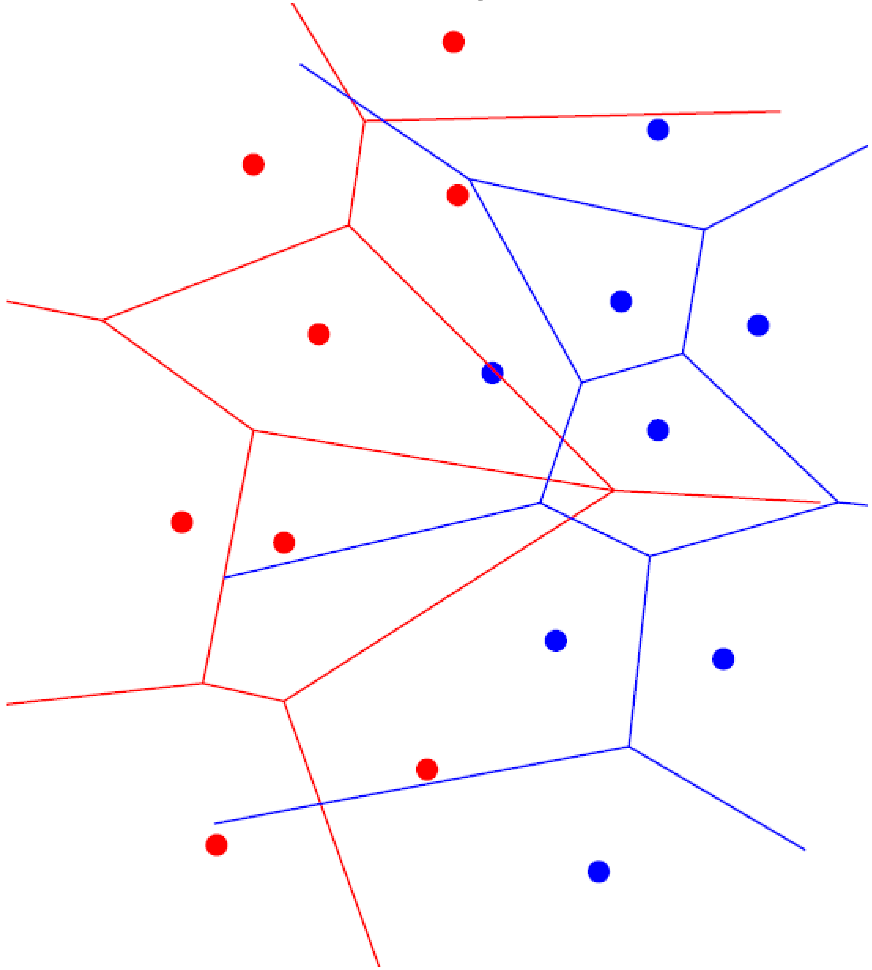
Divide and conquer

- We sort the input points and divide them vertically to two subsets *R* and *B* of approximately the same size



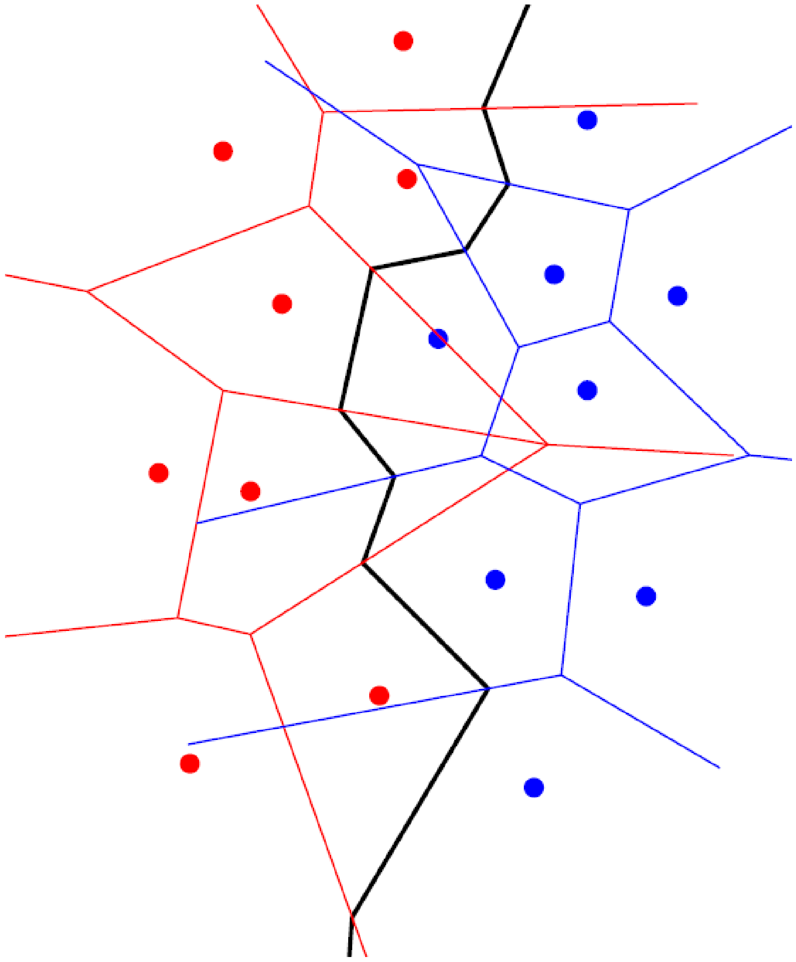
Divide and conquer

- We calculate recursively Vor(R) and Vor(B)



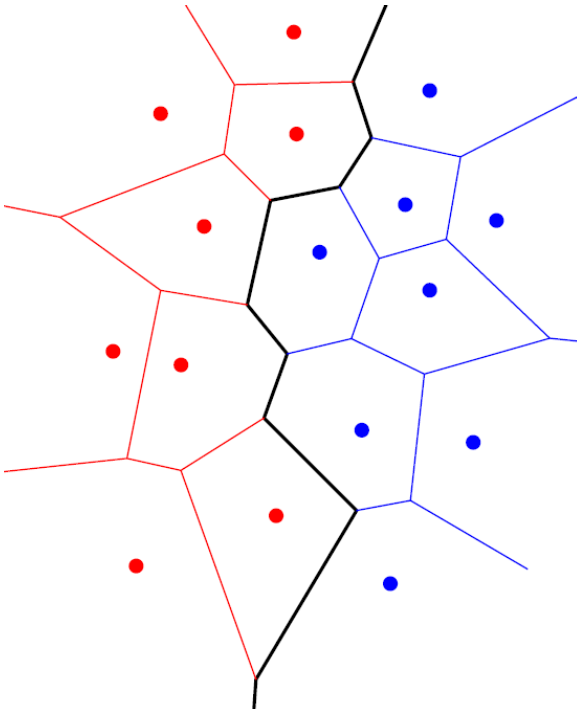
Divide and conquer

- We determine so called **separating chain**



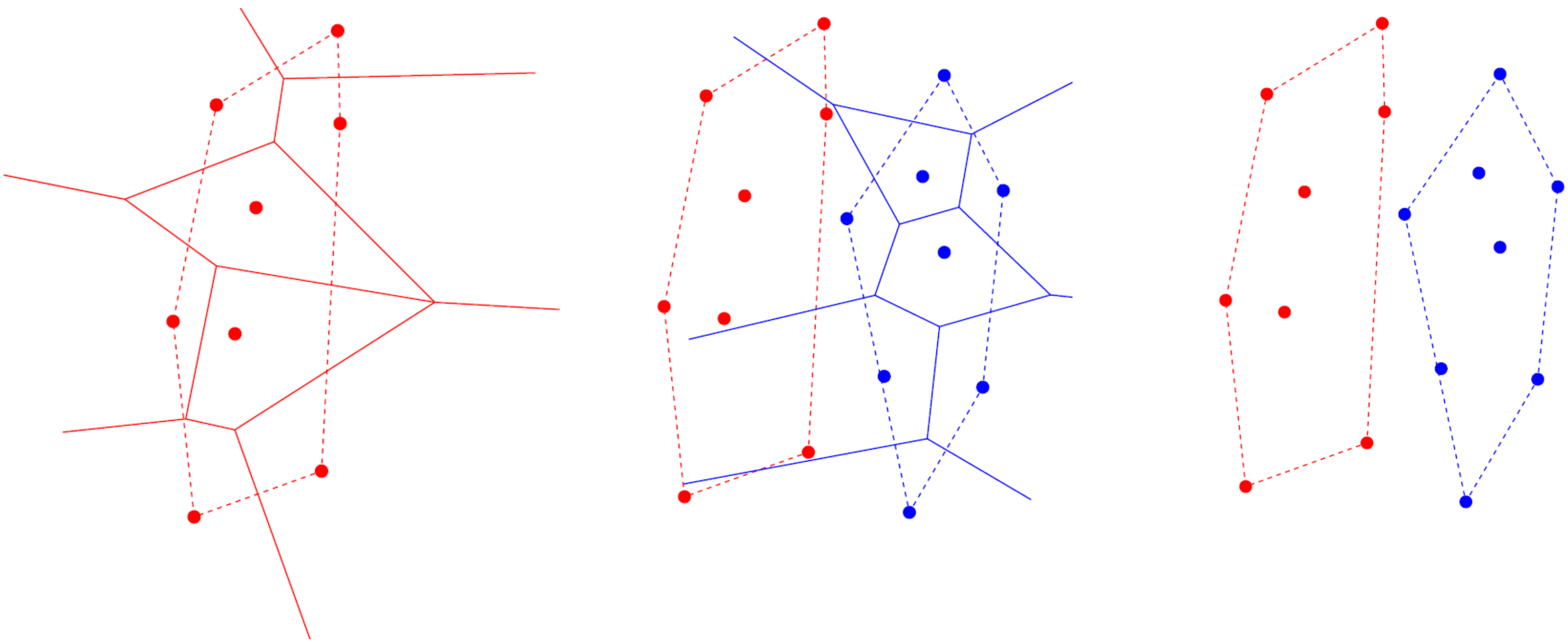
Divide and conquer

- We remove the part of Vor(R) lying on the right side from the separating chain and the par of Vor(B) lying on the left side from the separating chain



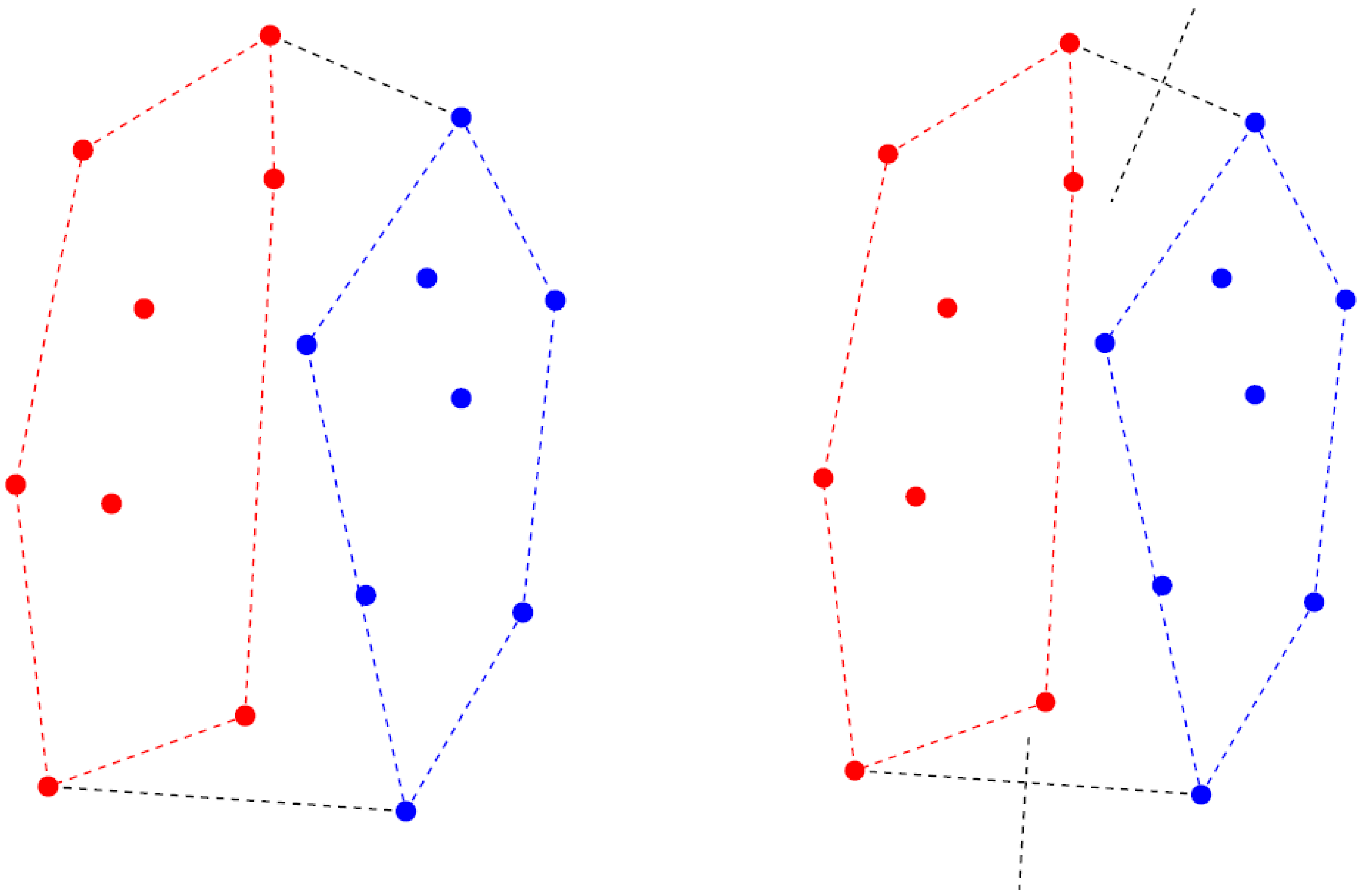
Divide and conquer

- Defining the separating chain:
  - First, we find two convex hulls …



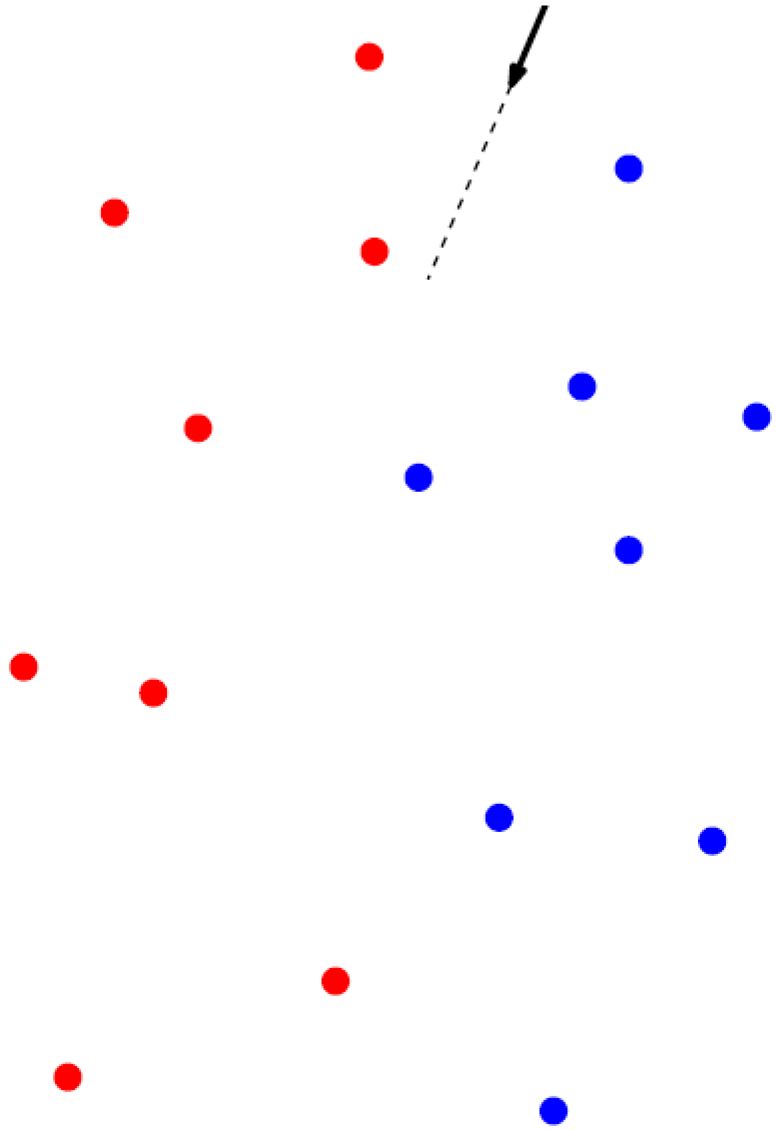
Divide and conquer

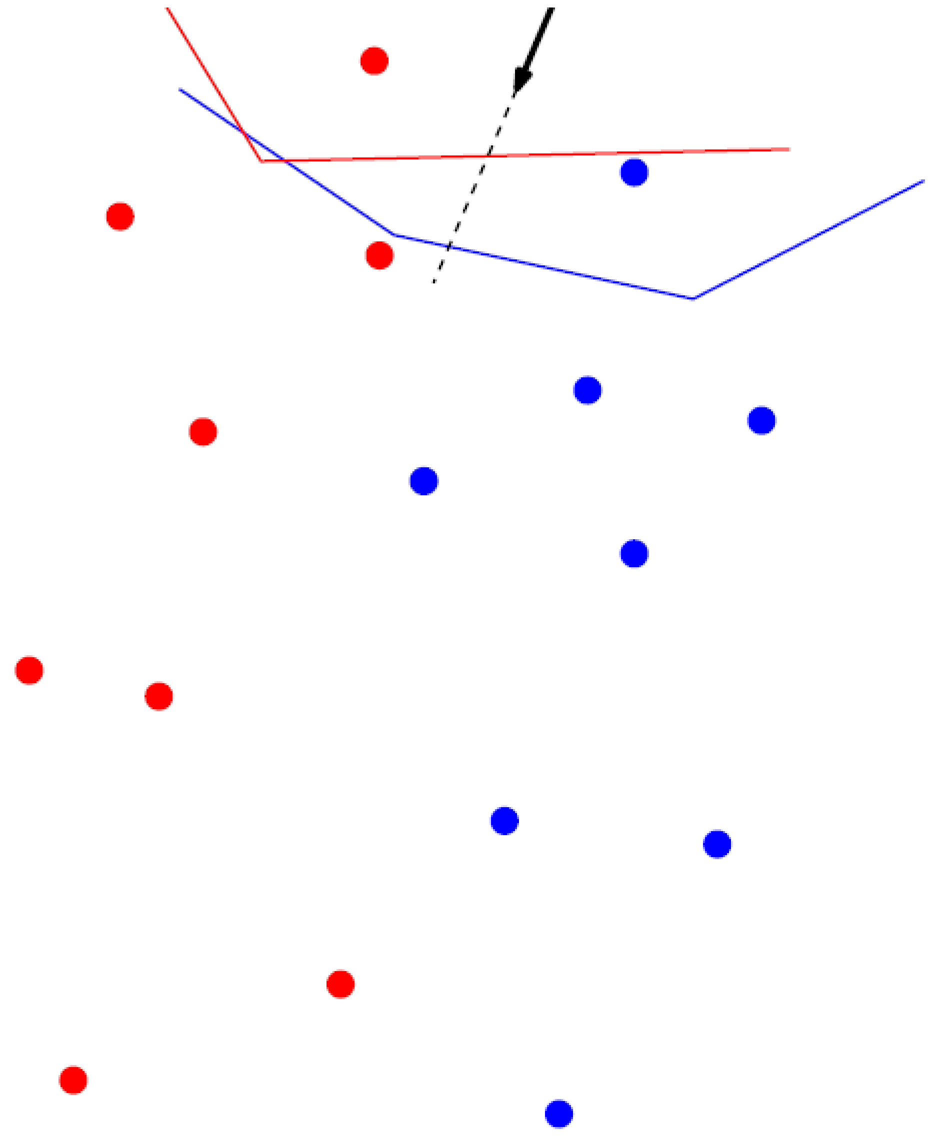
… and the upper and lower tangent and their perpendicular half-lines

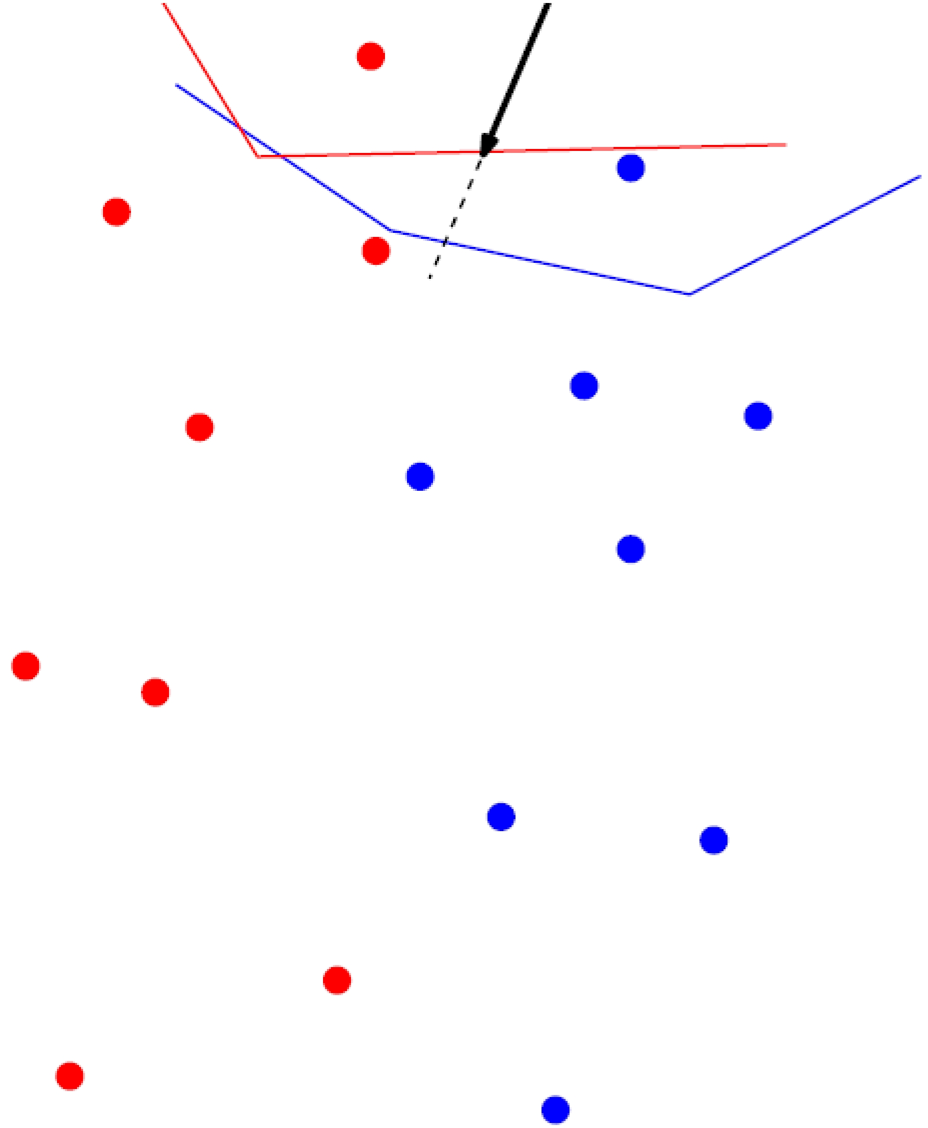


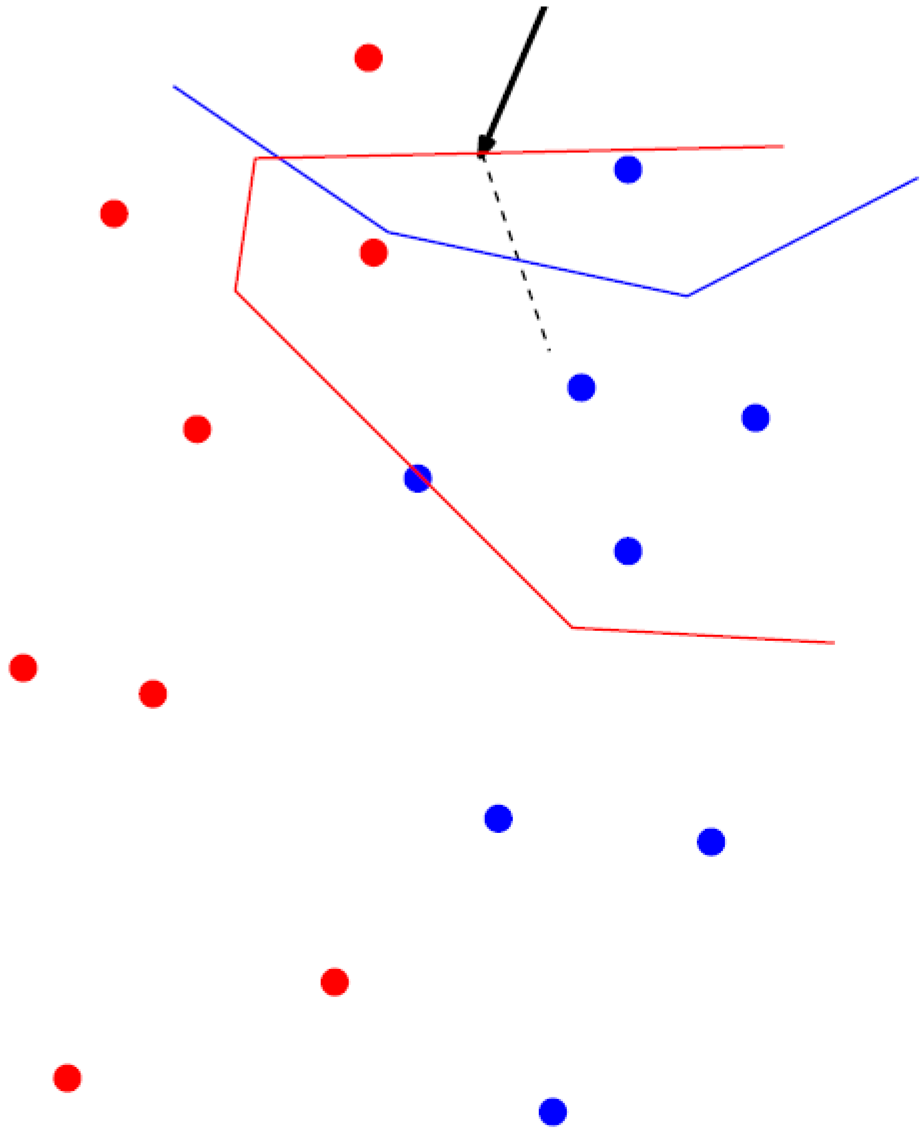
Divide and conquer

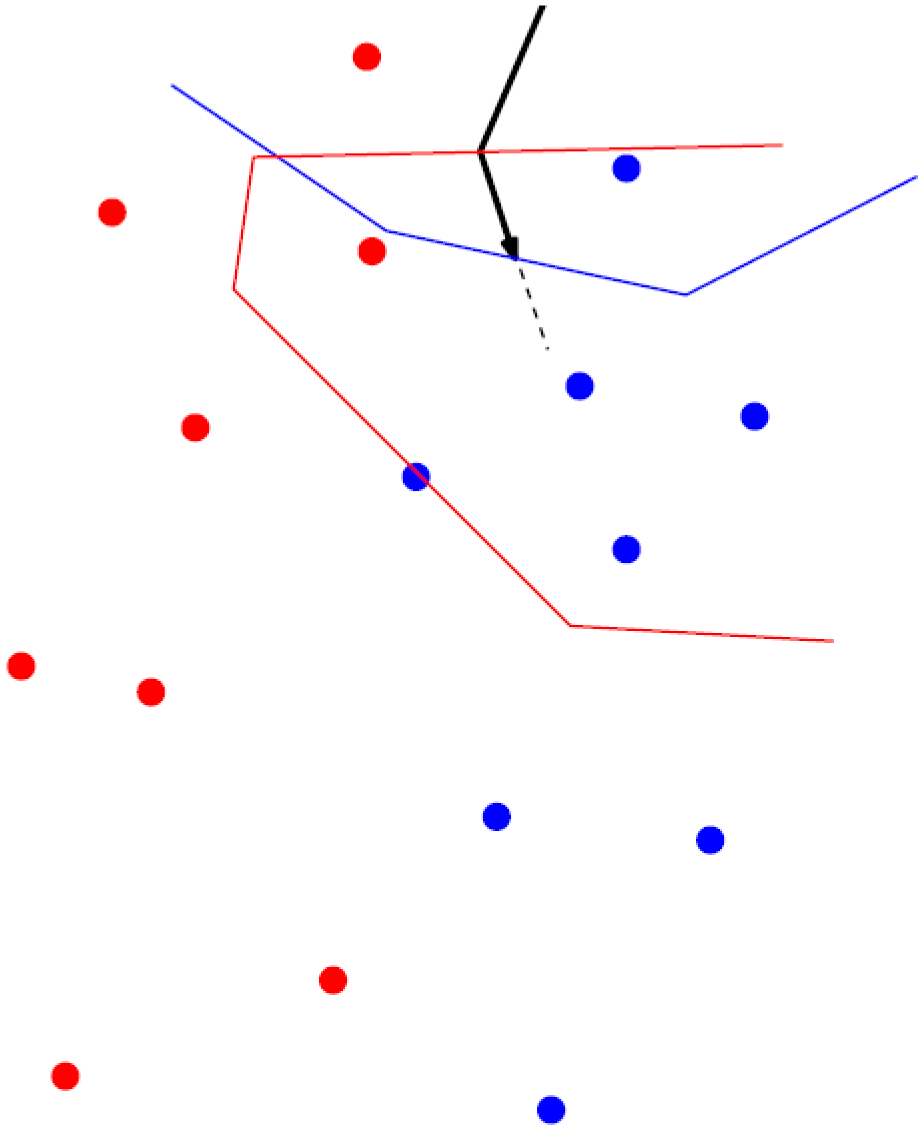
- We start from one of these half-lines and continue with the following procedure, until we reach the second half-line:
  - Always when there starts an edge $e \in b(R, B)$ for which $e \subset b_{ij}$, $p_i \in R$, $p_j \in B$:
    - Search for the intersection of edge e with $\mathrm{Vor}_{\mathrm{R}}(p_i)$
    - Search for the intersection of edge e with $\mathrm{Vor}_{\mathrm{B}}(p_j)$
    - Select one of these intersections
    - Determine p_k corresponding to a new starting region
    - Replace p_i or p_j (according to the selected point) by new p_k
    - Repeat this step with the new edge

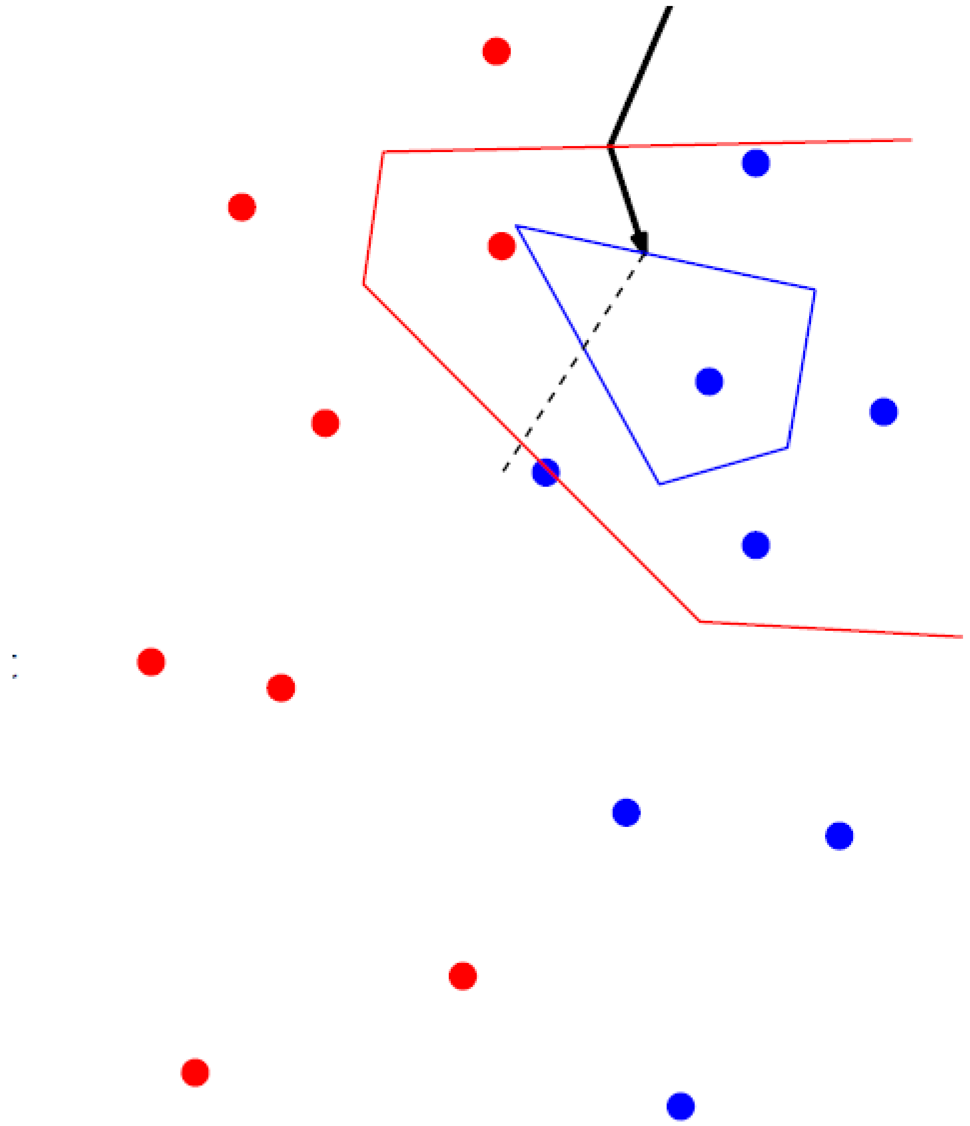


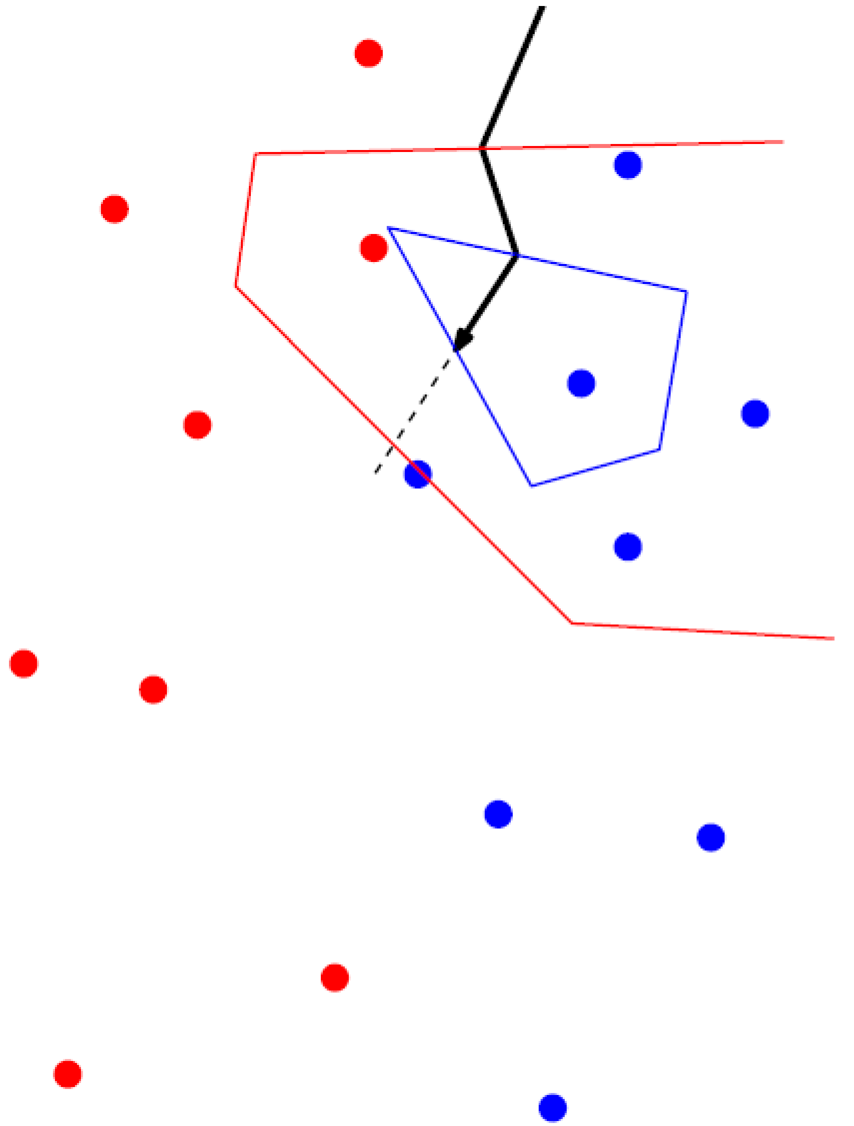


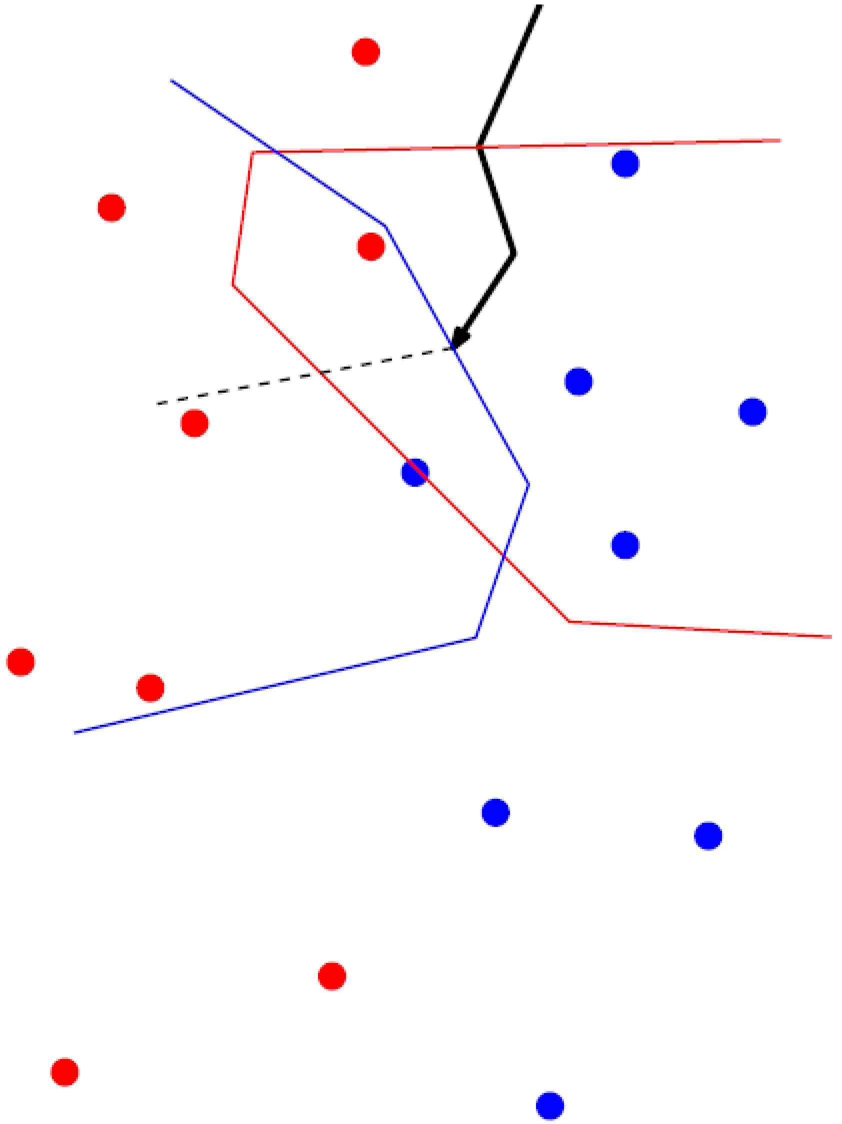


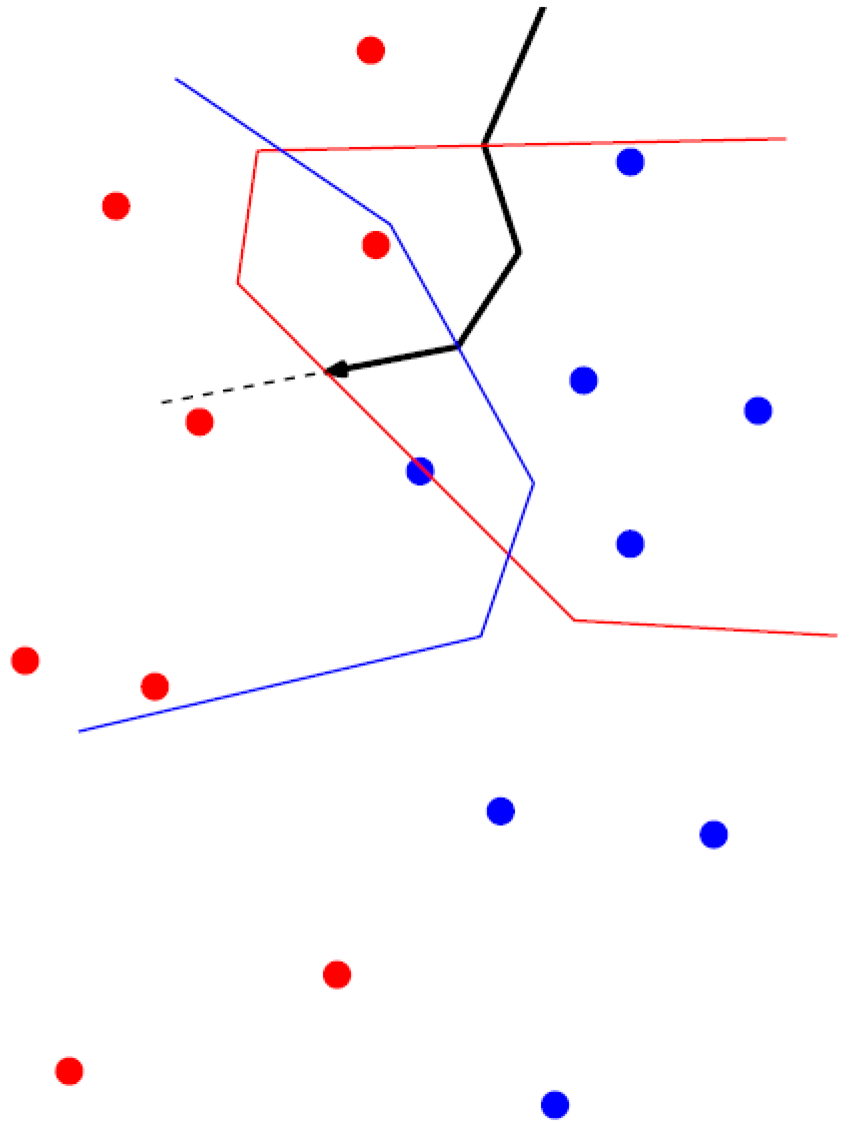


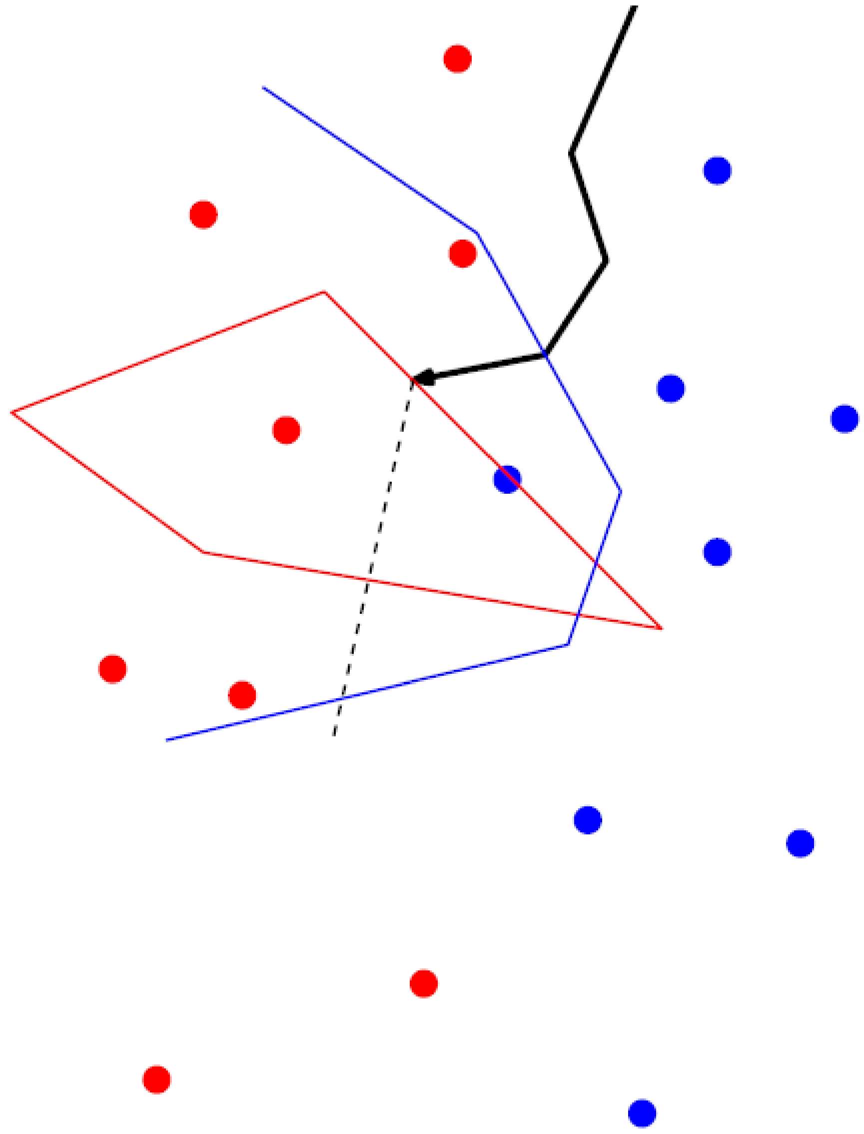


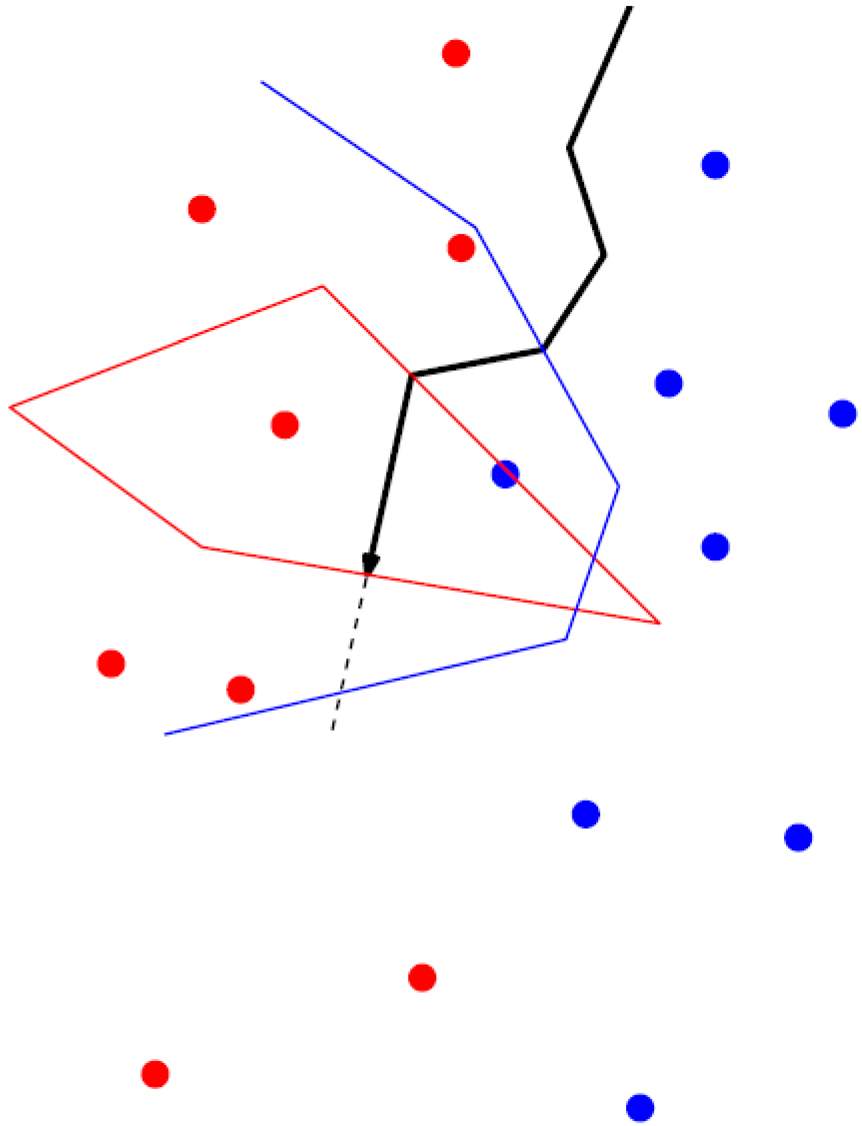


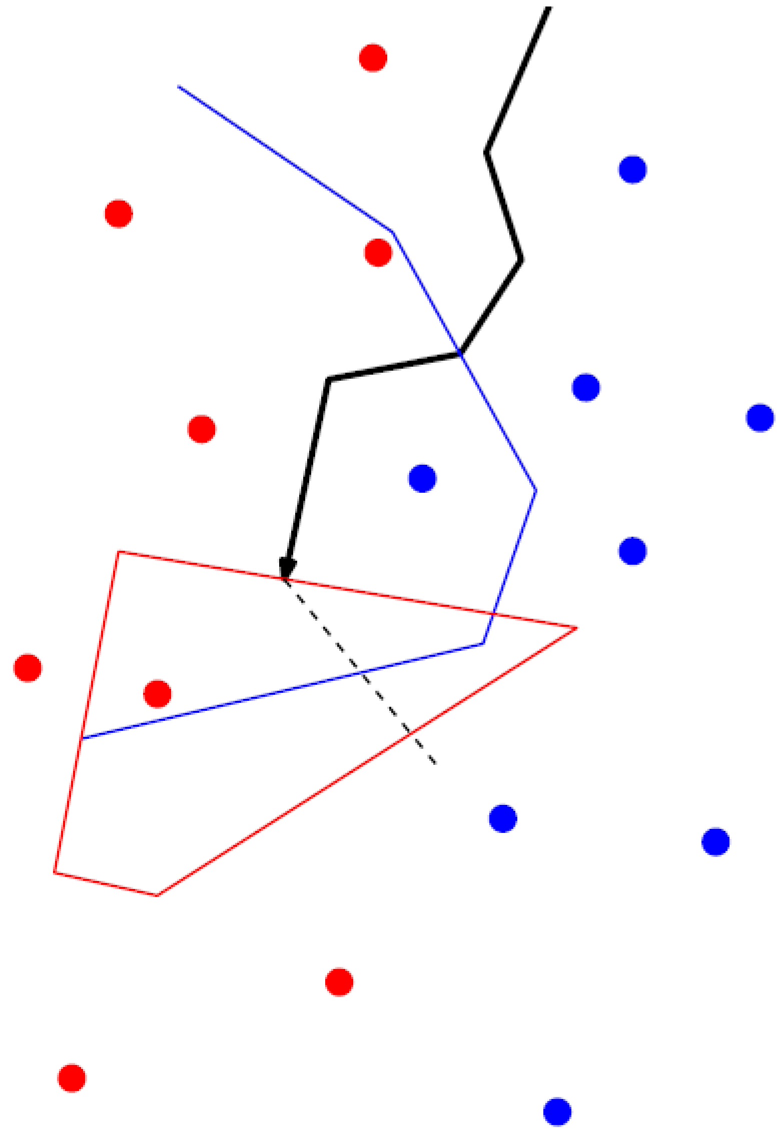


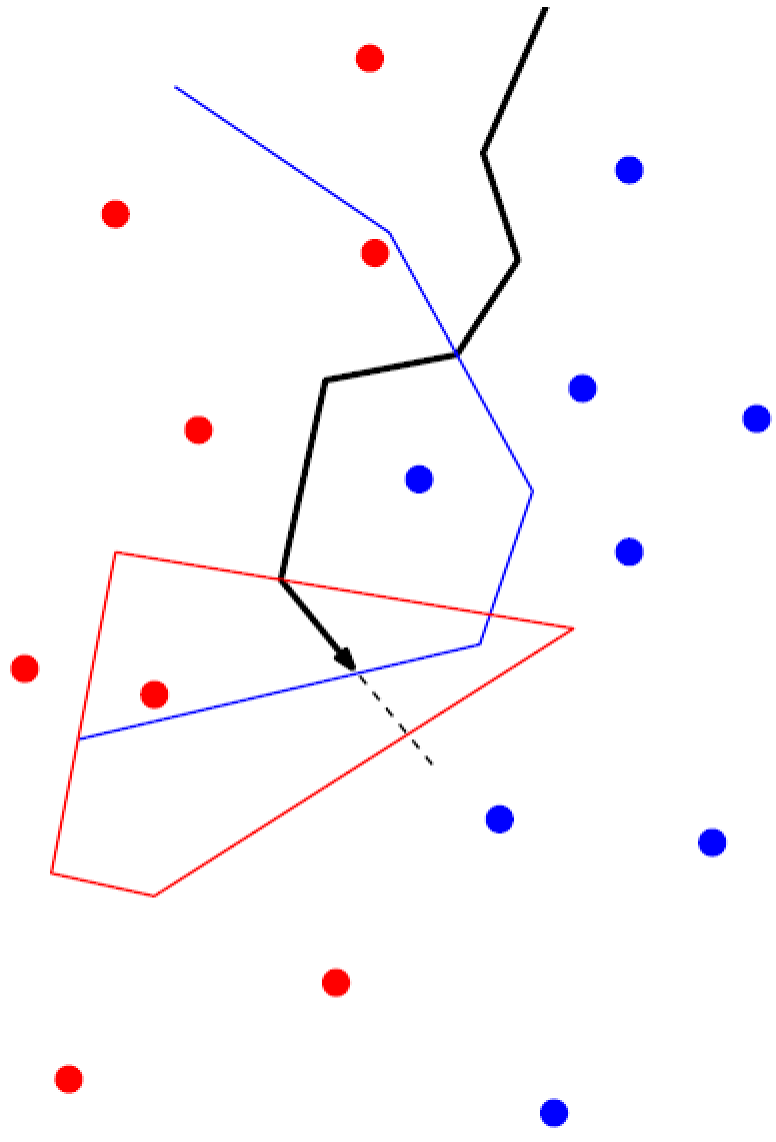


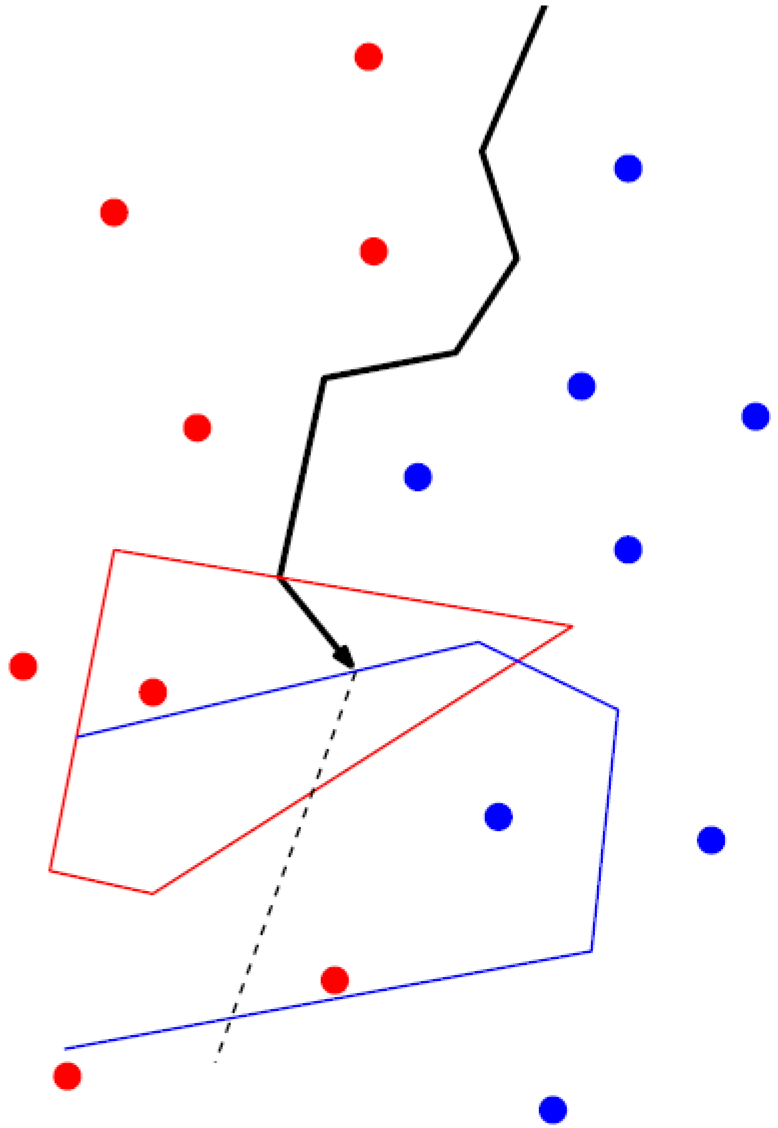


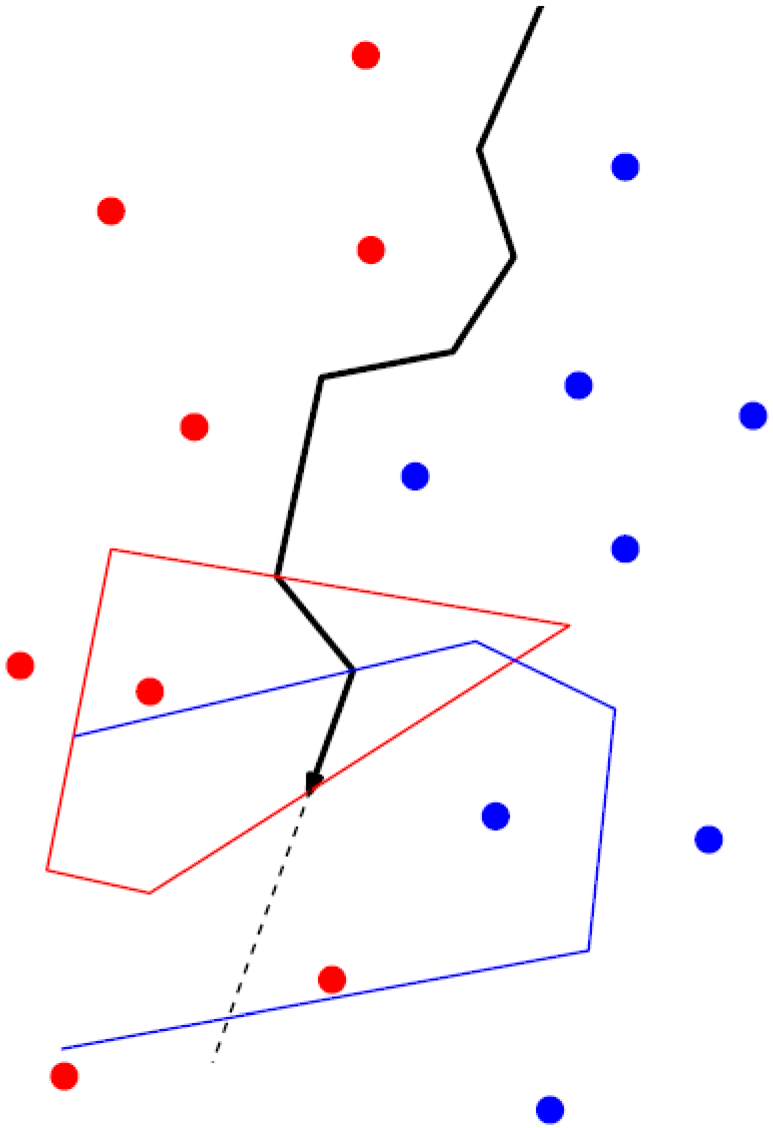


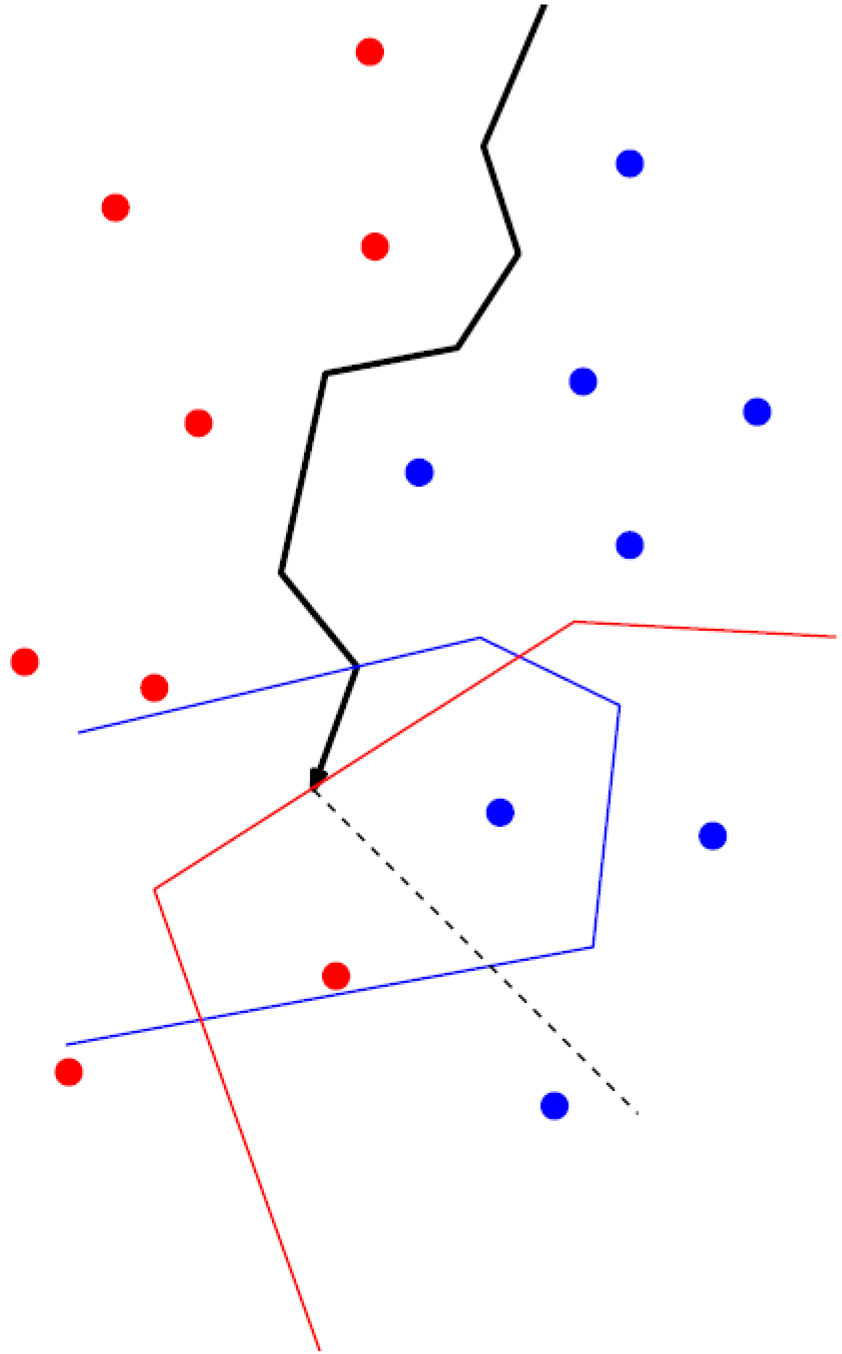


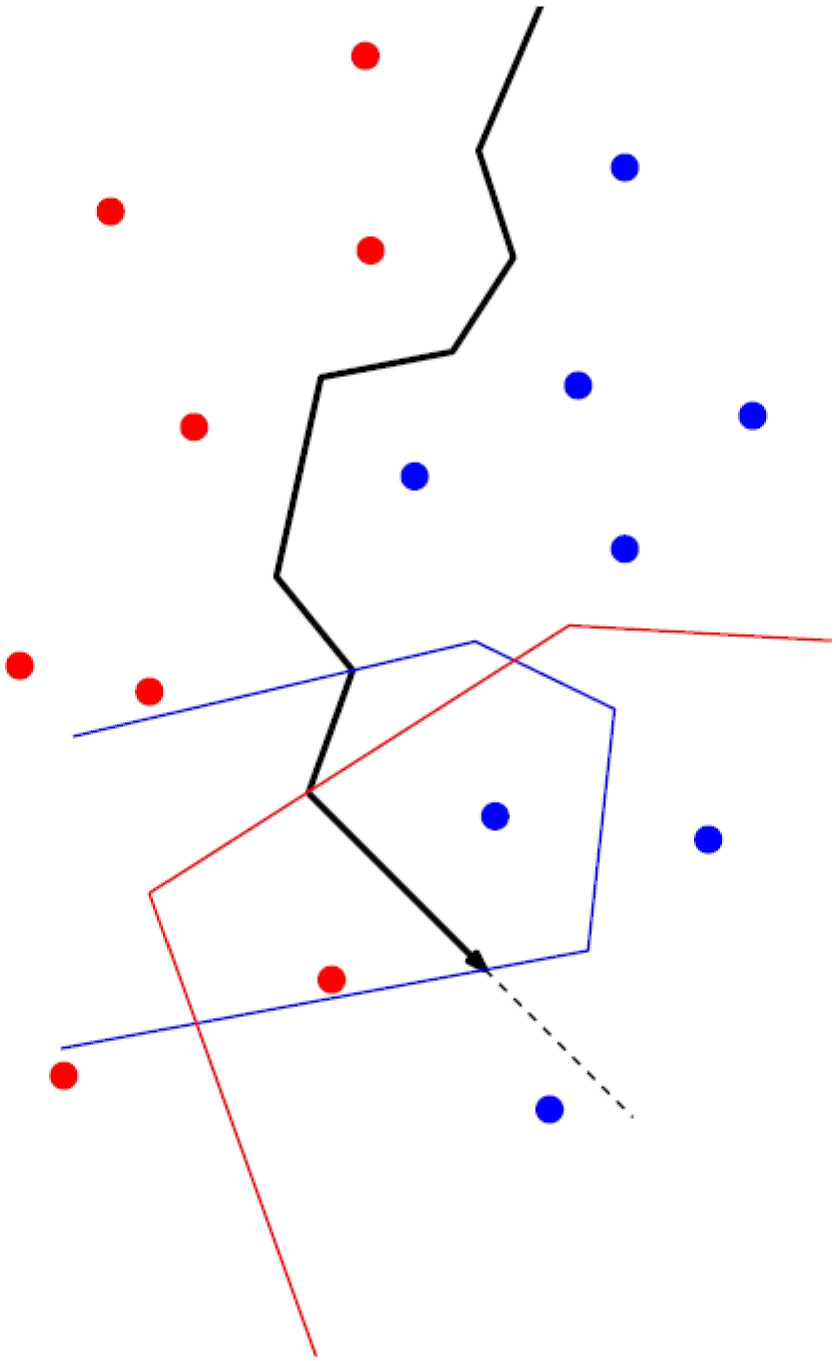


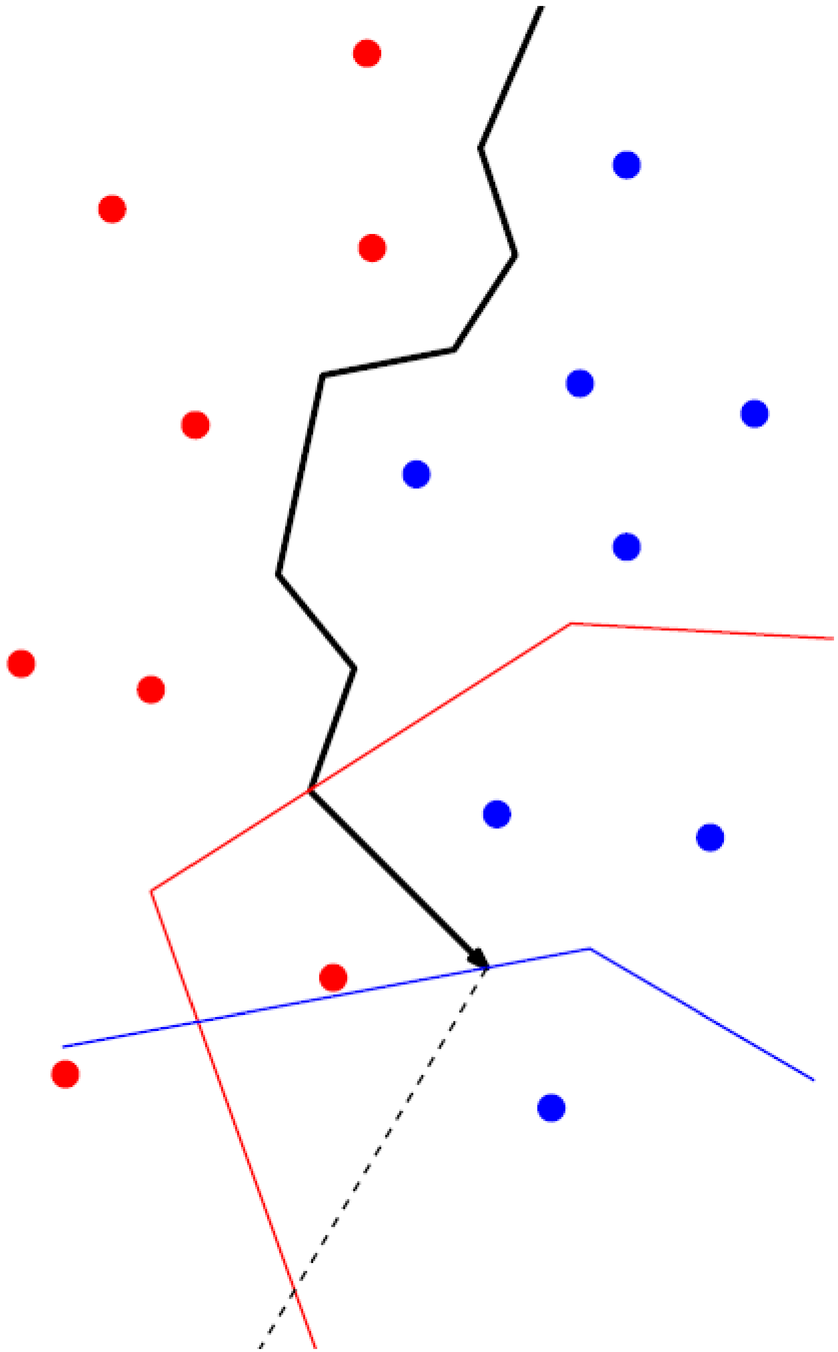


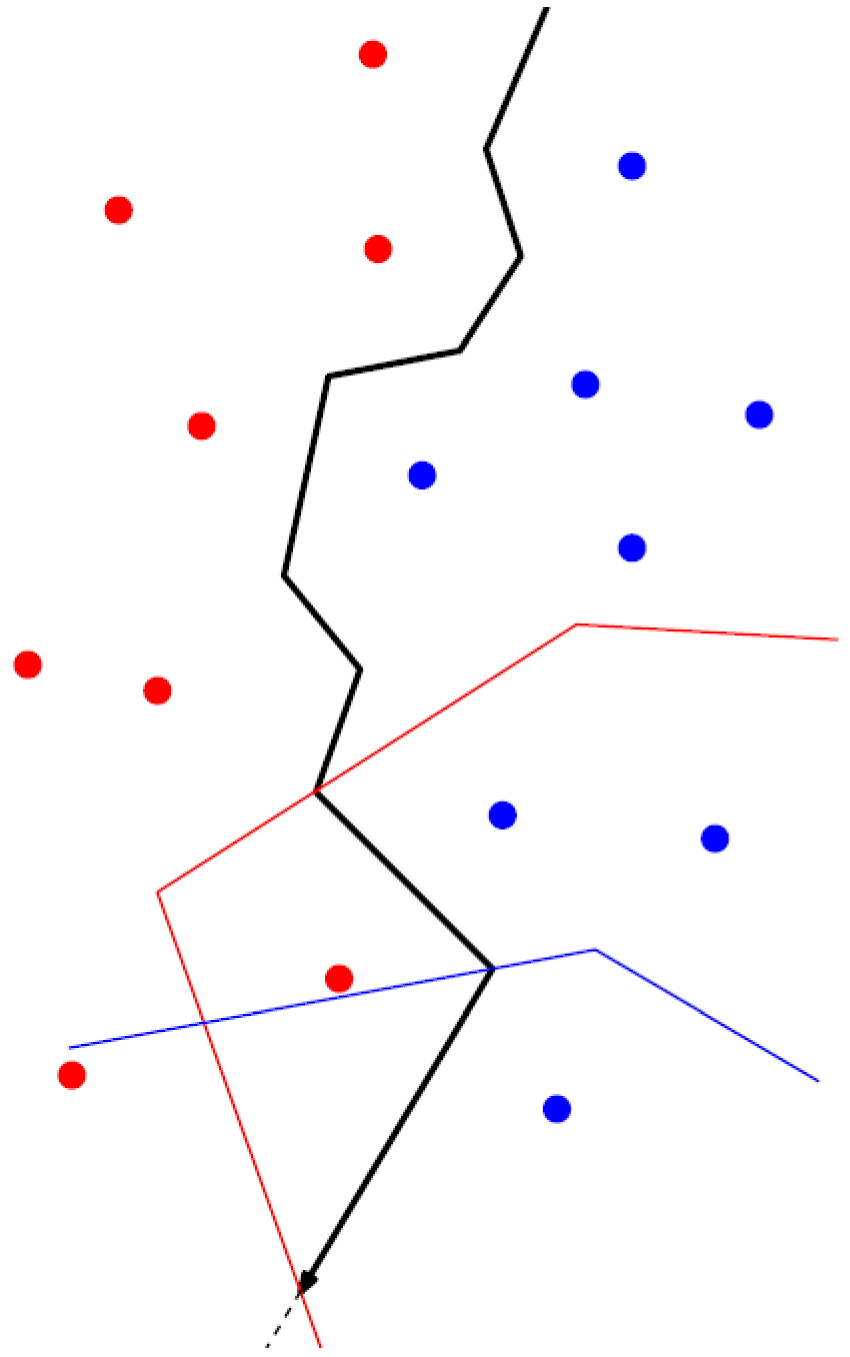


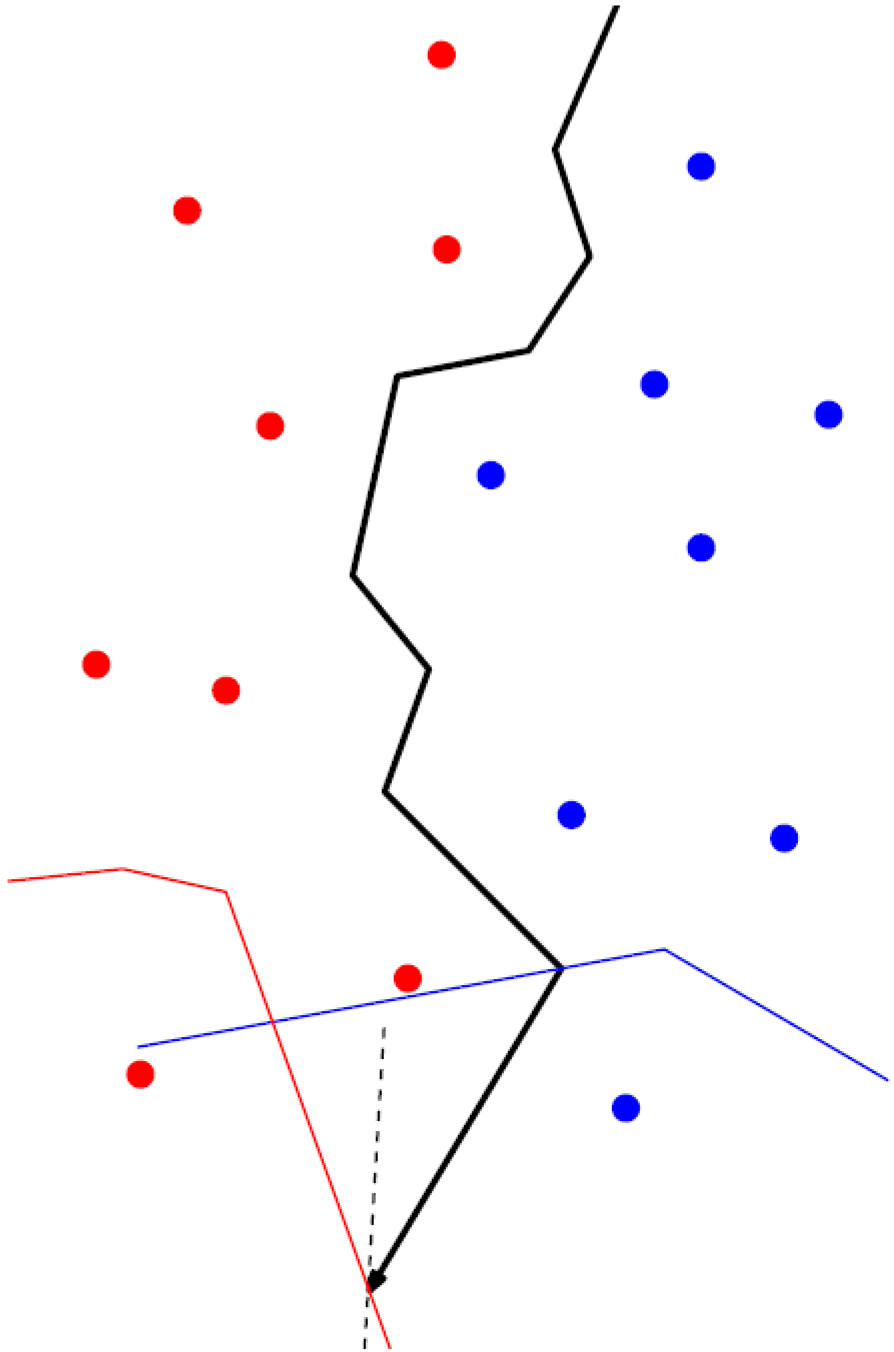


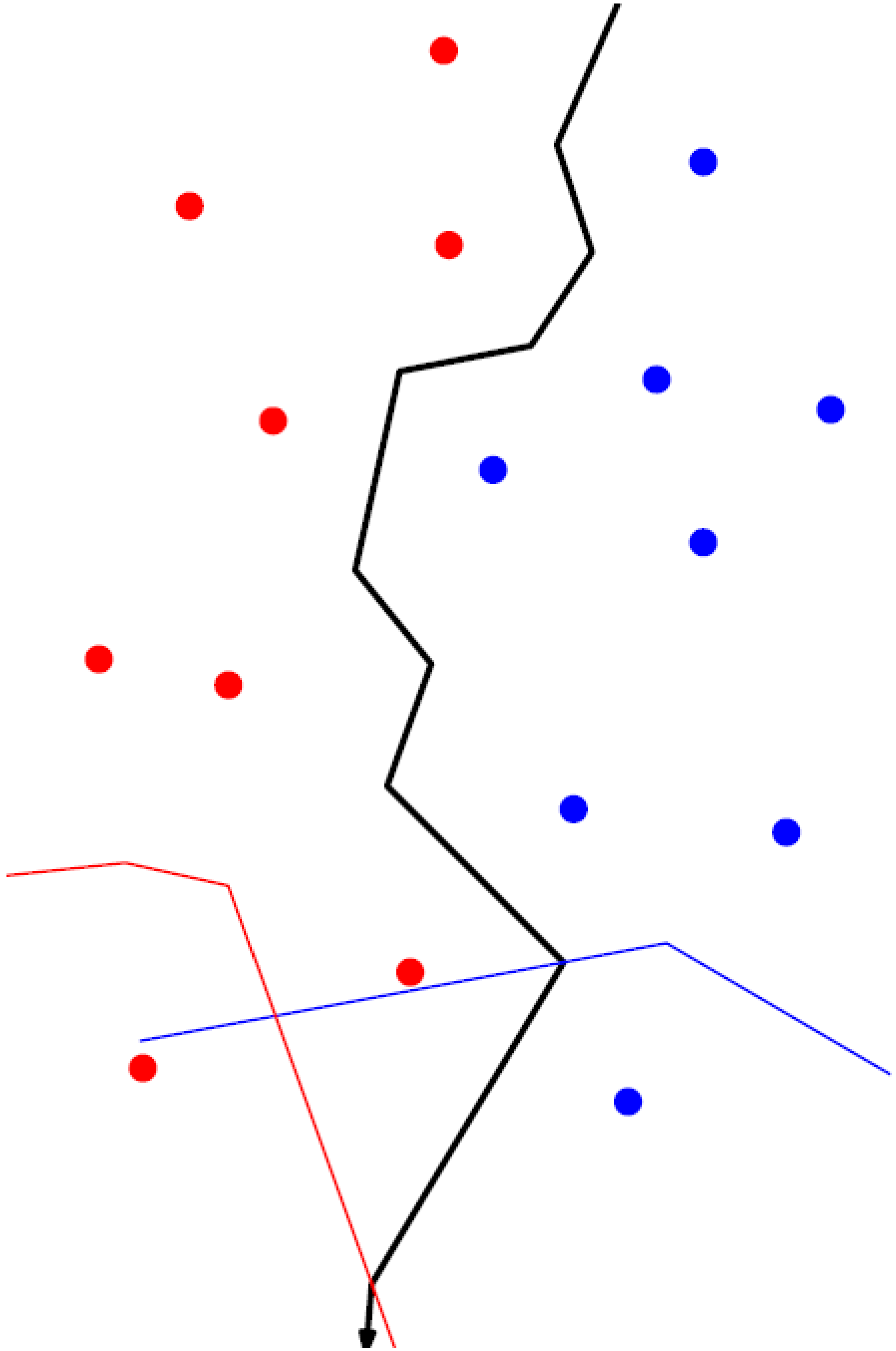


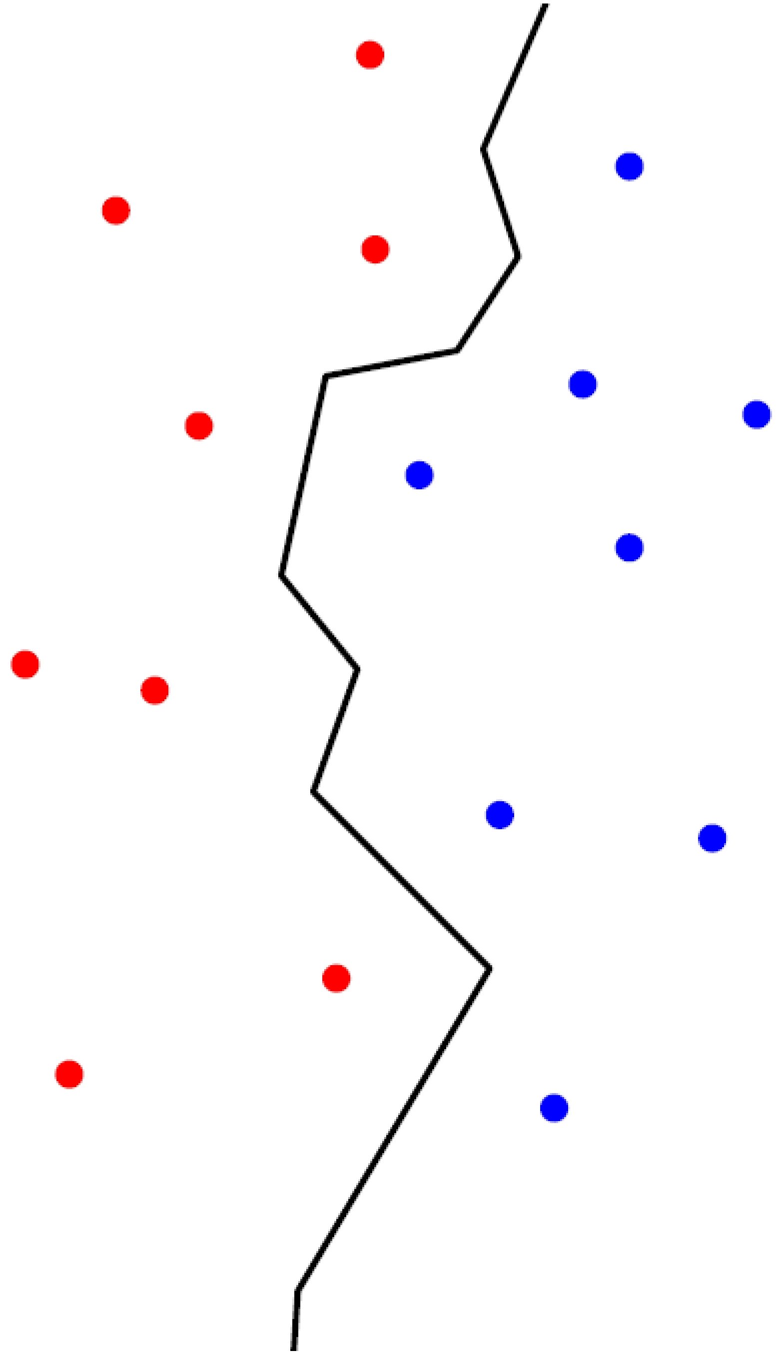












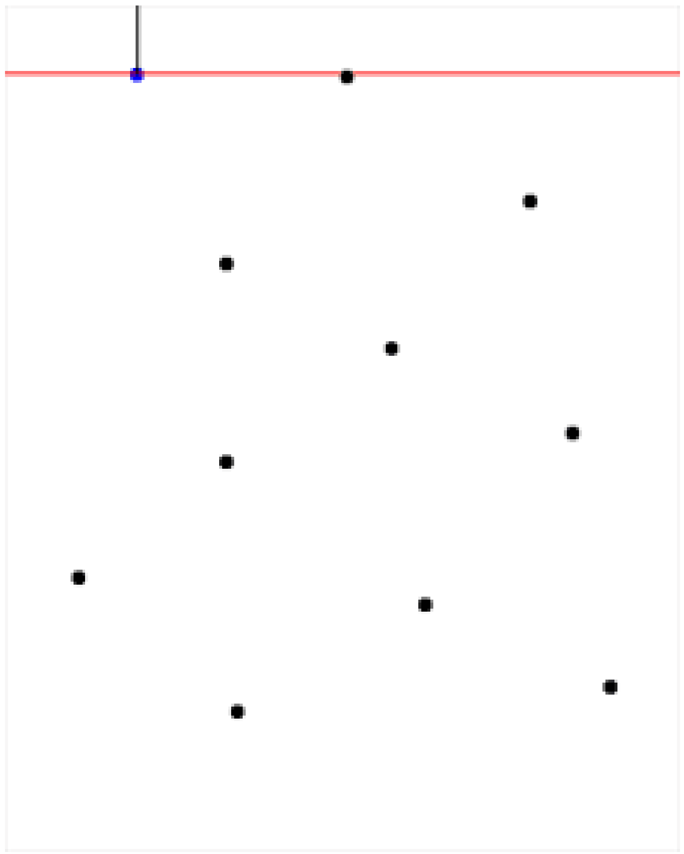
Sweep line (Fortune's algorithm)

- Algorithm uses so-called „sweep line“ and „beach line“, both of them traversing the space containing the input points
- The sweep line can be horizontal or vertical, heading from top to bottom or vice versa
- Invariant of the algorithm = for the input points already traversed by the sweep line we have already a correct VD constructed, the rest of the points was not processed yet

Sweep line (Fortune’s algorithm)

- „Beach line“ is not in fact a line but a curve above the sweep line, consisting of parts of parabolas
- A set of all points being closer to some of the points above the sweep line than to the sweep line itself is delineated by parabolic arcs – their connection forms the beach line

Sweep line (Fortune's algorithm)



Sweep line (Fortune's algorithm)

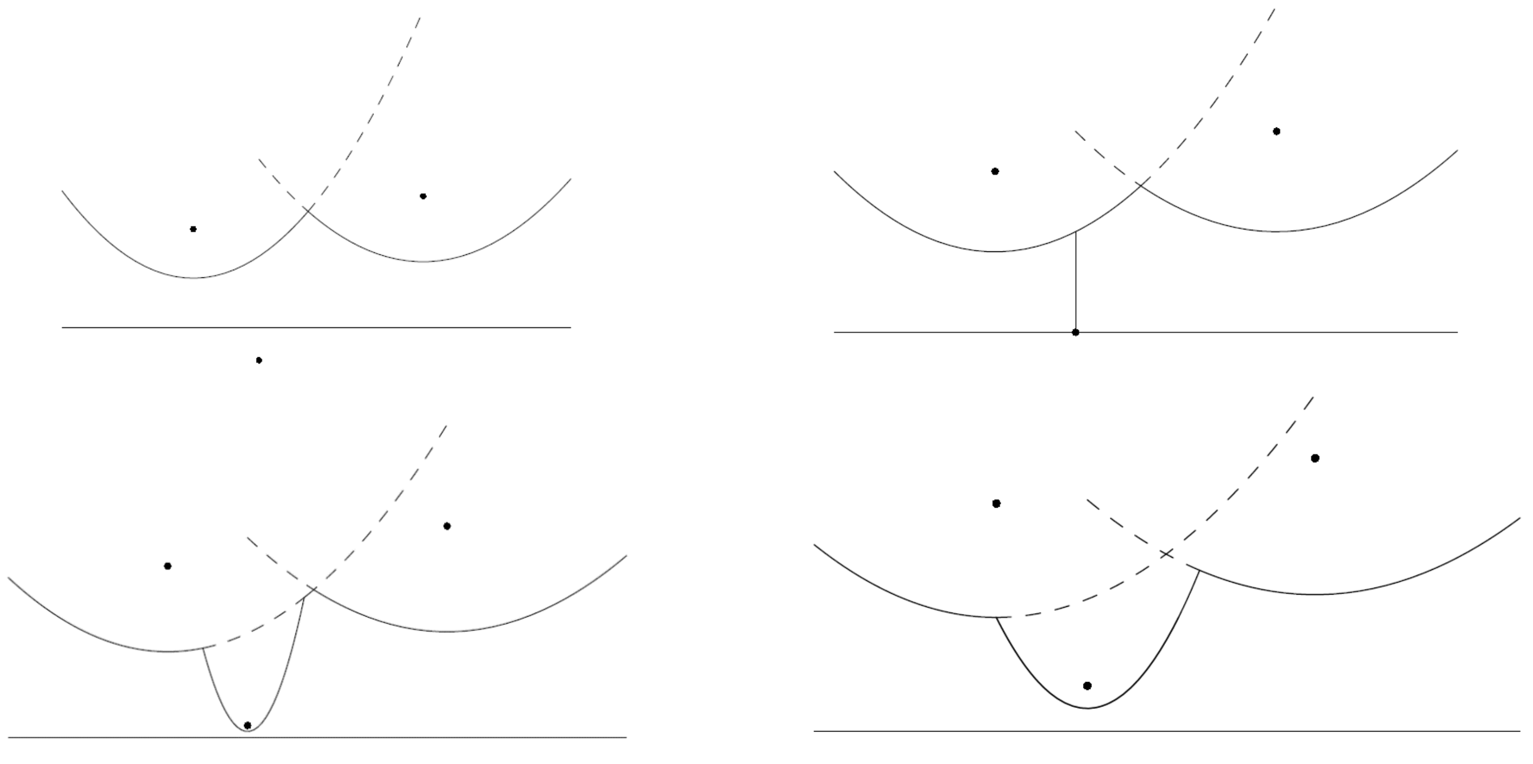
- The intersection of arcs lying on the beach line lie on the edges of the VD. With moving the sweep line, these intersections create the edges of VD Vor(*P*)
- The algorithms contains the following two operations:

Sweep line (Fortune's algorithm)

- **Site event** – a new generating point emerges on the beach line, we have to add it to the VD structure
- **Circle event** – when one of the parabolic arcs is terminated

Site event

- This event generates a new parabolic arc on the beach line and its intersection with the current beach line starts to create a new VD edge



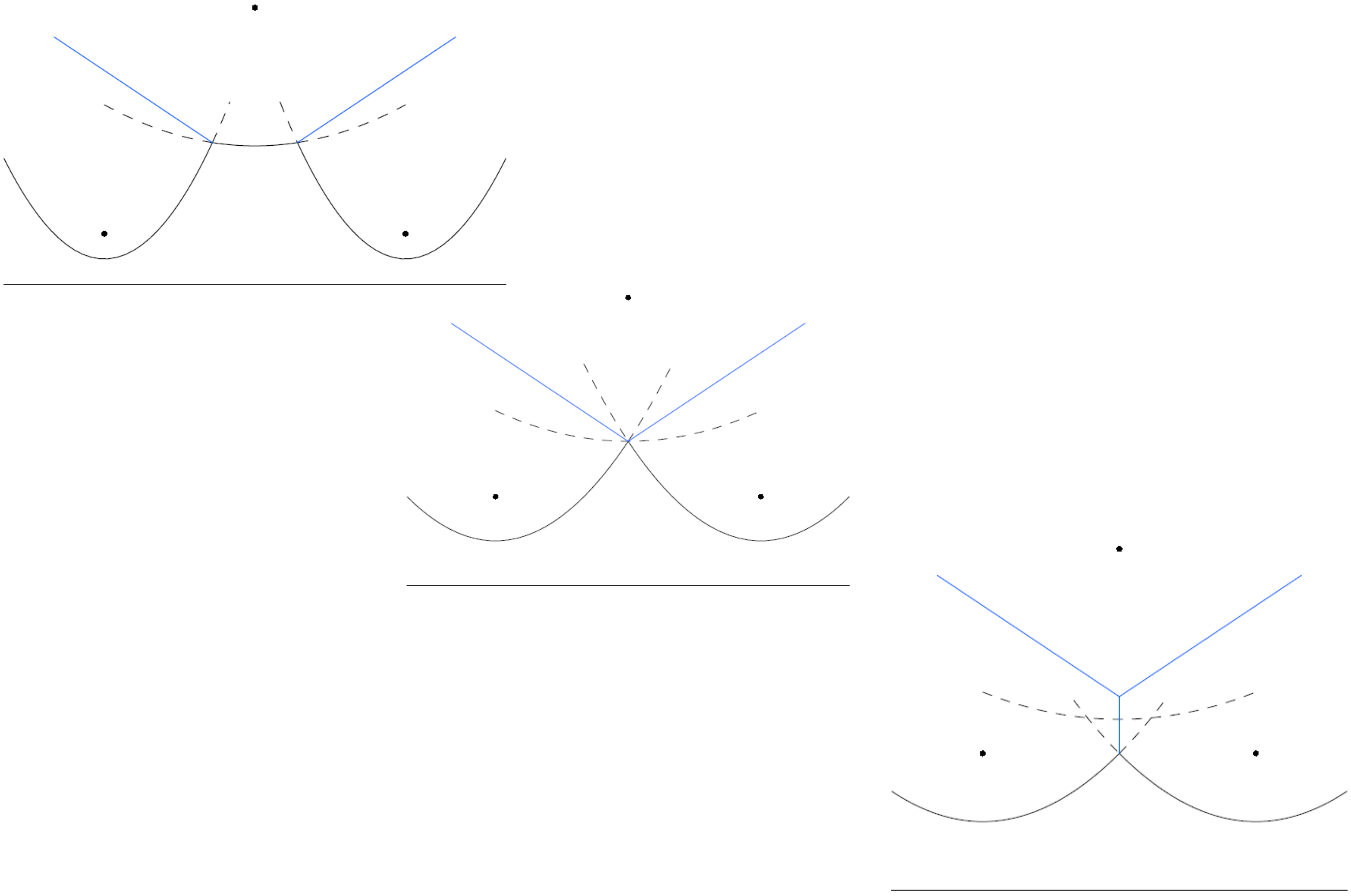
Site event

- Beach line consists of maximally $2n - 1$ parabolic arcs, because each generating point creates one parabola and divides maximally one existing parabolic arc to two parts

Circle event

- When some of the parabolic arcs is terminated
- This happens when three parabolas generated by points P_i, P_j, P_k all intersect in point Q – then this point Q forms the new Voronoi vertex

Circle event



Sweep line (Fortune's algorithm)

- More information, details for implementation:
  - http://blog.ivank.net/fortunes-algorithm-and-implementation.html

Weighted Voronoi diagrams

- One of possible generalizations of VD, when each generating point is assigned to a weight. This weight influences the size and shape of the VD cell.
- Lets assign weight $w_i \in R$ to point P_i. Then we define the corresponding metrics as

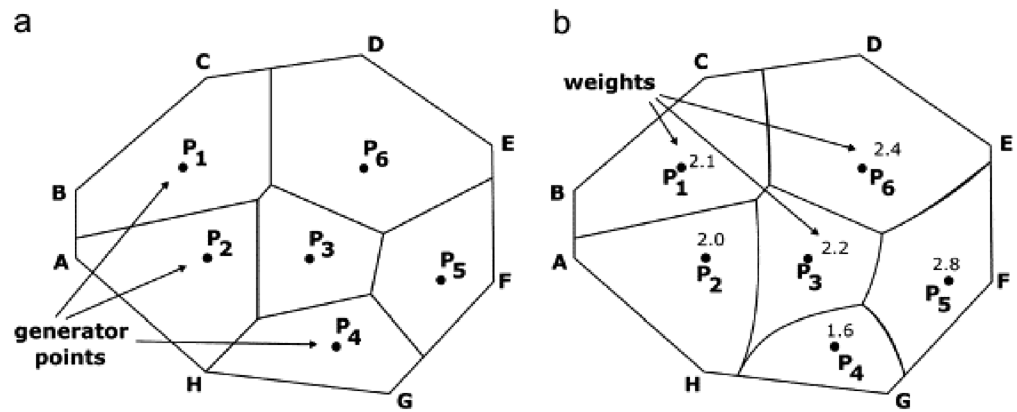
$$\mathrm{dist}_{\mathrm{WVD}}(P, Q) = \mathrm{dist}(P, Q) - w_i$$

where dist can be an arbitrary metrics

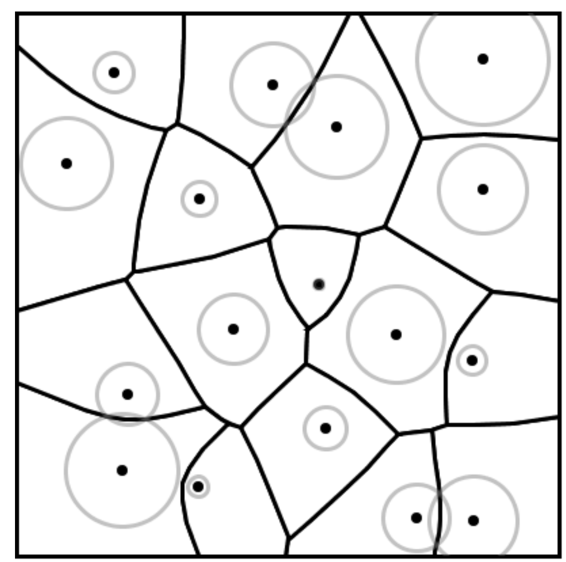
Weighted Voronoi diagrams

- When increasing the weight of a given point the corresponding VD cell is increasing which correspond to the given metric
- When the dist metric is the Euclidean distance, then $dist_{WVD}(P, P_i)$ can be interpreted as the distance of point P from a circle with center in P_i and radius w_i
- Voronoi edges are in this case parts of hyperbolas

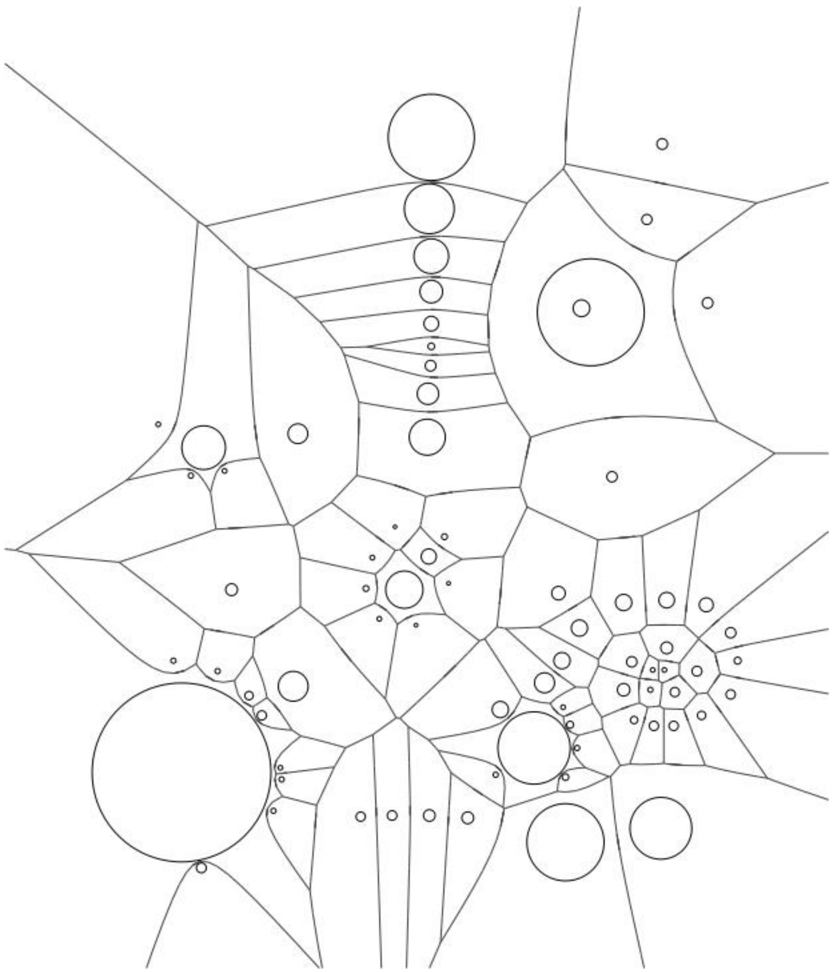
Weighted Voronoi diagrams



www.sciencedirect.com



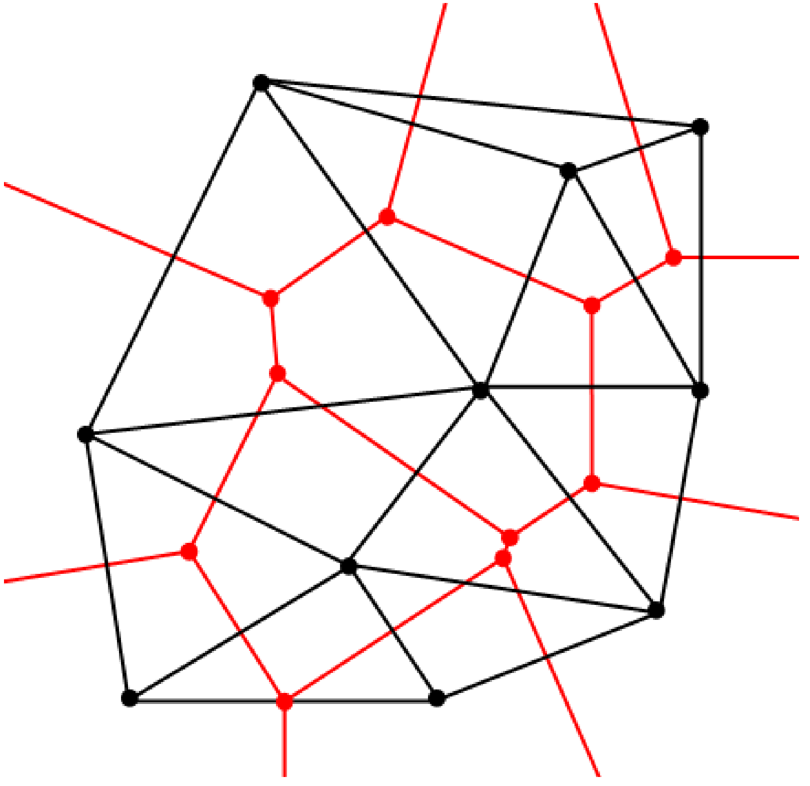
d.hatena.ne.jp



www.cgal.org

Assignment

- Use the already constructed Delaunay triangulation for the construction of Voronoi diagram
- Visualize it



fr.wikipedia.org