# IA008: Computational Logic

## 5. Inductive Inference

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# Basic Concepts

# Induction

learning **general facts** from **examples:**

**Induction** is the process of forming of a **hypothesis** (about a **target concept/function**) based on **observed data.**

**Example**

What is the next number?

0, 1,

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0, 1, 1, 2,

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learning **general facts** from **examples:**

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0, 1, 1, 2, 3,

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learning **general facts** from **examples:**

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**Example**

What is the next number?

0, 1, 1, 2, 3, 5,

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What is the next number?

0, 1, 1, 2, 3, 5, 8,… $a_n = a_{n-2} + a_{n-1}$

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What is the next number?

0, 1, 1, 2, 3, 5, 8,… $a_n = a_{n-2} + a_{n-1}$

0, 0, 0, 0, 0, 0,… $a_n = 0$

or

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What is the next number?

0, 1, 1, 2, 3, 5, 8,… $a_n = a_{n-2} + a_{n-1}$

0, 0, 0, 0, 0, 0,… $a_n = 0$

or $a_n = n(n-1)(n-2)(n-3)(n-4)(n-5)$?

# Fundamental Problem

From a strictly logical point of view, induction is **not possible:** there are always several possible explanations for the observed phenomena and there is no rational basis for choosing one over the others. Hence, a hypothesis can be **falsified** but never **verified.**

Consequently we need to make **additional a priori assumptions** (the so-called **inductive bias**) regarding the target concept.

### Inductive Learning Hypothesis

A hypothesis that approximates the target concept well over a sufficiently large amount of training data will also approximate it well over unobserved examples.

### Occam's Razor

Use the **simplest** hypothesis that matches the observations.

(What's simple depends on our formalism.)

# Philosophy of Science

## Scientific Method

In the 17th century, **Francis Bacon, René Descartes,** and **Isaac Newton** developed the **scientific method** based on induction.

## Problem of Induction

**David Hume** was the first to point out that inductive inferences are unprovable and always subject to falsification.

## Falsifiability

**Karl Popper** argued that induction does not exist. Instead science is based on **conjecture** and **criticism.** One should select hypotheses that are the easiest to falsify.

## Paradigm Shift

**Thomas Kuhn** viewed science as a **social process.** He emphasised the role of **paradigms** and the way they are replaced when sufficiently many observations point to problems with the current paradigm.

# Machine Learning

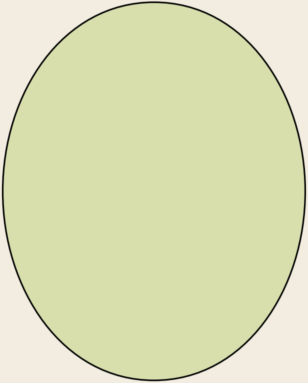
Induction (and learning in general) works best if it is **interactive:**

- form a hypothesis based on the **current data**
- test the hypothesis on **new data**
- repeat

The question therefore is not whether the hypothesis is **true,** but **how well** it predicts observations.

Most decent algorithms for inference use **statistical methods** and fall outside the scope of this course.

# Hypothesis Space



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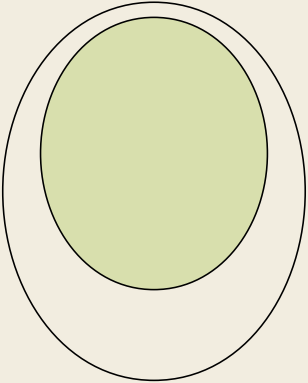

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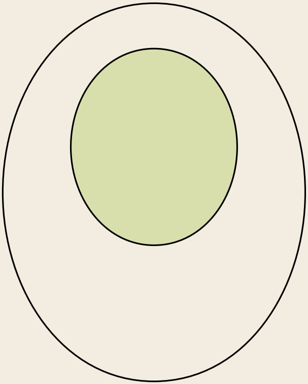

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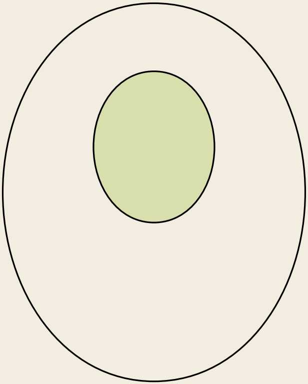

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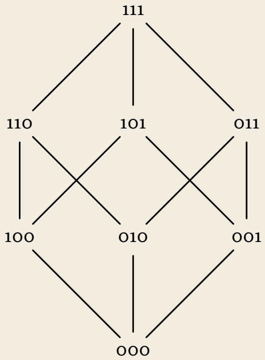

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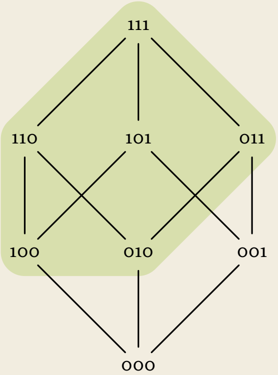
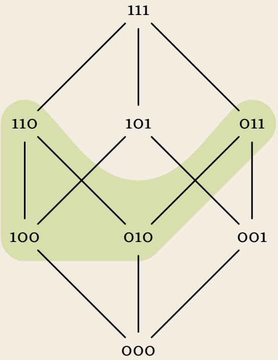



$x \vee y$

# Example




$x \vee y$

$\neg x \vee \neg z$

# Boolean Functions

# Boolean functions

In this lecture we will concentrate on learning **boolean functions**

$f : \{0,1\}^n \to \{0,1\}$

(which can be encoded as propositional formulae)

**Example**

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | $f(\bar{x})$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | √ |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | × |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | × |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | √ |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | √ |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | × |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | √ |

# Conjunctive Hypotheses

**Setting**

Learning a boolean function $f : \{0, 1\}^n \to \{0, 1\}$ using as hypotheses **conjunctions** $\eta := x_i \wedge \cdots \wedge \neg x_k$ of literals.

**General-to-specific ordering**

$\eta$ is **more specific** than $\zeta$ if $\eta \vDash \zeta$.

**Idea**

Find the **most specific** hypothesis.

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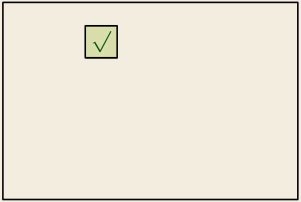

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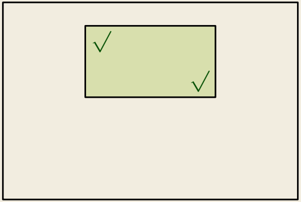

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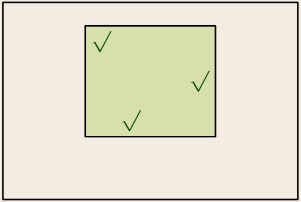

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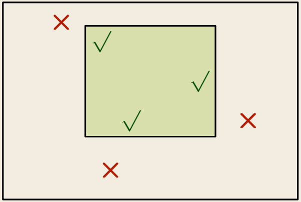

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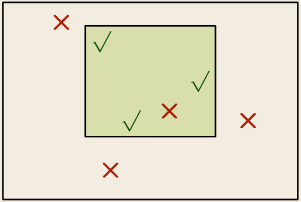

# Find-S algorithm

- Start with $\eta := \bot$
- Consider the next positive example $\bar{b}$
- If $\eta(\bar{b})$ is true, continue.
- Otherwise, find the most specific $\zeta$ such that $\eta \vDash \zeta$ and $\zeta(\bar{b})$ is true.
- Continue with $\eta := \zeta$.

This algorithm computes find the least conjunction with respect to the $\vDash$-ordering that covers all positive examples.
If any of the negative examples is also covered, the training data cannot be described by a conjunction.

# Example

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | $f(\bar{x})$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | √ |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | × |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | × |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | √ |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | √ |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | × |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | √ |

$\eta_0 := \bot$

# Example

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|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | $\surd$ |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | × |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | × |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | $\surd$ |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | $\surd$ |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | × |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | $\surd$ |

$\eta_0 := \bot$

$\eta_1 := \neg x_1 \wedge x_2 \wedge \neg x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge x_{10}$

# Example

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | $f(\bar{x})$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | ✓ |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | ✗ |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | ✗ |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | ✓ |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | ✓ |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | ✗ |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | ✓ |

$\eta_0 := \bot$

$\eta_1 := \neg x_1 \wedge x_2 \wedge \neg x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge x_{10}$

$\eta_2 := \neg x_1 \wedge \neg x_3 \wedge x_5 \wedge \neg x_8$

# Example

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|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | ✓ |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | ✗ |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | ✗ |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | ✓ |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | ✓ |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | ✗ |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | ✓ |

$\eta_0 := \bot$

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$\eta_2 := \neg x_1 \wedge \neg x_3 \wedge x_5 \wedge \neg x_8$

$\eta_3 := \neg x_1 \wedge \neg x_3 \wedge x_5$

# Example

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | $f(\bar{x})$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | ✓ |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | ✗ |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | ✗ |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | ✓ |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | ✓ |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | ✗ |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | ✓ |

$\eta_0 := \bot$

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$\eta_2 := \neg x_1 \wedge \neg x_3 \wedge x_5 \wedge \neg x_8$

$\eta_3 := \neg x_1 \wedge \neg x_3 \wedge x_5$

$\eta_4 := \neg x_1 \wedge \neg x_3 \wedge x_5$

# Hypothesis space

**Goal** Compute **all** hypotheses consistent with the data.

Let $D \subseteq \{0,1\}^n \times \{0,1\}$ be the observed data and $H$ the set of all hypotheses consistent with every datum in $D$.

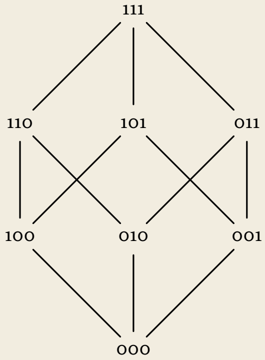
We compute the sets $H^+$ and $H^-$ of maximal/minimal elements of $H$ (with respect to the general-to-specific order $\vDash$).

**Candidate-Elimination Algorithm**

- Start with $H^+ := \{\top\}$ and $H^- := \{\bot\}$.
- For each positive $d \in D$:
  - Delete from $H^+$ every hypothesis $\eta$ with $\eta(d) = 0$.
  - Replace every $\eta \in H^-$ with $\eta(d) = 0$ by the set of all minimal $\zeta$ such that
    $$\eta \vDash \zeta\,, \quad \zeta(d) = 1\,, \quad \text{and} \quad \zeta \vDash \eta'\,, \quad \text{for some } \eta' \in H^+.$$
  - Remove from $H^-$ all elements that are not minimal.
- For each negative $d \in D$: proceed analogously with $H^+$ and $H^-$ interchanged.

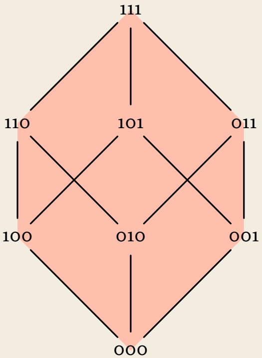
# Example

| $x_1$ | $x_2$ | $x_3$ | $f(\bar{x})$ |
|---|---|---|---|
| 1 | 1 | 0 | ✓ |
| 0 | 0 | 1 | ✗ |
| 1 | 0 | 0 | ✓ |
| 1 | 0 | 1 | ✗ |



# Example

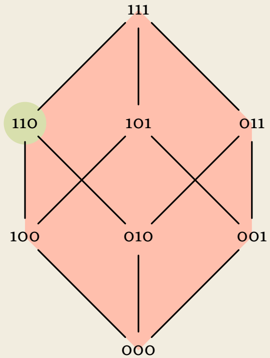
| $x_1$ | $x_2$ | $x_3$ | $f(\bar{x})$ |
|---|---|---|---|
| 1 | 1 | 0 | $\surd$ |
| 0 | 0 | 1 | $\times$ |
| 1 | 0 | 0 | $\surd$ |
| 1 | 0 | 1 | $\times$ |




Step 0. $H^- = \{\bot\}$ $H^+ = \{\top\}$

# Example

| $x_1$ | $x_2$ | $x_3$ | $f(\bar{x})$ |
|---|---|---|---|
| 1 | 1 | 0 | ✓ |
| 0 | 0 | 1 | ✗ |
| 1 | 0 | 0 | ✓ |
| 1 | 0 | 1 | ✗ |

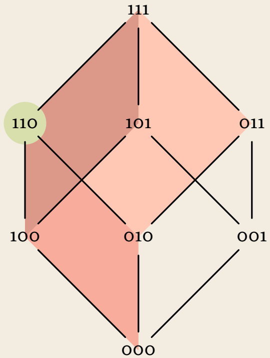



Step 0. $H^- = \{\bot\} \quad H^+ = \{\top\}$

Step 1. $H^- = \{x_1 \wedge x_2 \wedge \neg x_3\} \quad H^+ = \{\top\}$

# Example

| $x_1$ | $x_2$ | $x_3$ | $f(\bar{x})$ |
|---|---|---|---|
| 1 | 1 | 0 | ✓ |
| 0 | 0 | 1 | ✗ |
| 1 | 0 | 0 | ✓ |
| 1 | 0 | 1 | ✗ |




Step 0. $H^- = \{\bot\}$ $\quad H^+ = \{\top\}$

Step 1. $H^- = \{x_1 \land x_2 \land \neg x_3\}$ $\quad H^+ = \{\top\}$

Step 2. $H^- = \{x_1 \land x_2 \land \neg x_3\}$ $\quad H^+ = \{x_1,\ x_2,\ \neg x_3\}$

# Example

| $x_1$ | $x_2$ | $x_3$ | $f(\bar{x})$ |
|---|---|---|---|
| 1 | 1 | 0 | $\surd$ |
| 0 | 0 | 1 | $\times$ |
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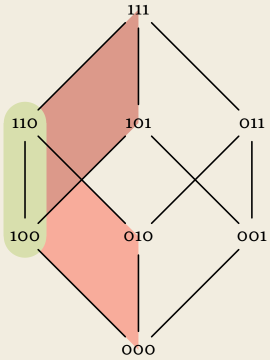


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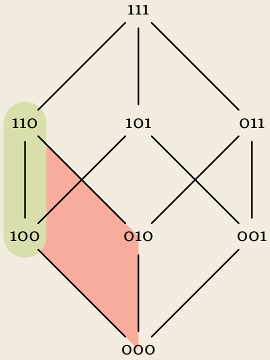
Step 3. $H^- = \{x_1 \wedge \neg x_3\}$ $H^+ = \{x_1,\ \neg x_3\}$

# Example

| $x_1$ | $x_2$ | $x_3$ | $f(\bar{x})$ |
|---|---|---|---|
| 1 | 1 | 0 | √ |
| 0 | 0 | 1 | × |
| 1 | 0 | 0 | √ |
| 1 | 0 | 1 | × |

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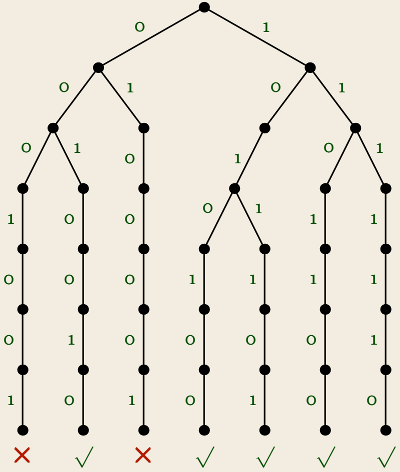
Step 4. $H^- = \{x_1 \wedge \neg x_3\}$ $H^+ = \{\neg x_3\}$

# Decision Trees

# Decision Trees

Organise the function to be learned as a tree.

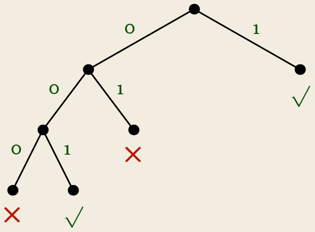
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $f(\bar{x})$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | √ |
| 0 | 1 | 0 | 0 | 0 | 1 | 1 | × |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | √ |
| 0 | 0 | 1 | 0 | 0 | 1 | 0 | √ |
| 0 | 0 | 0 | 1 | 1 | 0 | 1 | × |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 | √ |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | √ |



# Decision Trees

Organise the function to be learned as a tree.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $f(\bar{x})$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | √ |
| 0 | 1 | 0 | 0 | 0 | 1 | 1 | × |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | √ |
| 0 | 0 | 1 | 0 | 0 | 1 | 0 | √ |
| 0 | 0 | 0 | 1 | 1 | 0 | 1 | × |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 | √ |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | √ |




The order of the variables $x_i$ matters. Which one do we choose?

# Ordered Binary Decision Diagrams (OBDDs)

- data structure to compactly represent a boolean function
- the arguments are **ordered** $x_1, \ldots, x_n$
- the graph is **reduced:** merge isomorphic subgraphs and eliminate unneeded vertices

$$(x_1 \wedge x_3) \vee (x_2 \wedge x_3) \vee \neg(x_1 \vee x_2 \vee x_3)$$