

IAoo8: Computational Logic

7. Many-Valued Logics

Achim Blumensath

blumens@fi.muni.cz

Faculty of Informatics, Masaryk University, Brno

Basic Concepts

Many-valued logics: Motivation

To some sentences we cannot – or do not want to – assign a truth value since

- ▶ they make **presuppositions** that are not fulfilled

John regrets beating his wife.

John does not regret beating his wife.

- ▶ they refer to **non-existing** objects

The king of Paris has a pet lion.

- ▶ they are too **vague**

The next supermarket is far away.

- ▶ we have **insufficient information**

The favourite colour of Odysseus was blue.

- ▶ we cannot determine their truth

The Goldbach conjecture holds.

This leads to logics with **truth values** other than ‘true’ and ‘false’.

3-valued logic

truth values 'false' \perp , 'uncertain' u , and 'true' \top .

A	$\neg A$	\wedge	\perp	u	\top	\vee	\perp	u	\top
\perp	\top	\perp	\perp	\perp	\perp	\perp	\perp	u	\top
u	u	u	\perp	u	u	u	u	u	\top
\top	\perp	\top	\perp	u	\top	\top	\top	\top	\top

Kleene K_3

\rightarrow	\perp	u	\top
\perp	\top	\top	\top
u	u	u	\top
\top	\perp	u	\top

Łukasiewicz L_3

\rightarrow	\perp	u	\top
\perp	\top	\top	\top
u	u	\top	\top
\top	\perp	u	\top

Example

A	B	$A \wedge (A \rightarrow B)$	$A \wedge (A \rightarrow B) \rightarrow B$
\perp	\perp	\perp	\top
\perp	u	\perp	\top
\perp	\top	\perp	\top
u	\perp	u	u
u	u	u	u/\top
u	\top	u	\top
\top	\perp	\perp	\top
\top	u	u	u/\top
\top	\top	\top	\top

Fuzzy logic

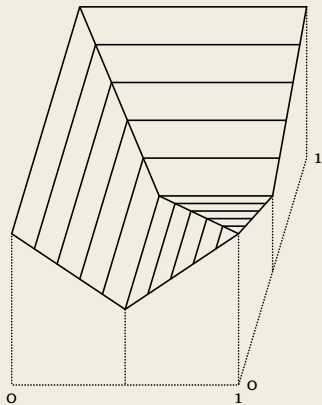
Truth values: $v \in [0, 1]$ measuring **how true** a statement is.
0 means 'false' and 1 means 'true'.

Several possible semantics:

$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$
$1 - A$	$A \cdot B$	$1 - (1 - A)(1 - B)$	$1 - A(1 - B)$
$1 - A$	$\min(A, B)$	$\max(A, B)$	$\max(1 - A, B)$
$1 - A$	$\max(A + B - 1, 0)$	$\min(A + B, 1)$	$\min(1 - A + B, 1)$

Example

$$A \wedge (A \rightarrow B) \rightarrow B = \max(1 - \min(A, \max(1 - A, B)), B)$$



Tableaux for L_3

statements: $t \leq \varphi$, $\varphi \leq t$, $t \not\leq \varphi$, or $\varphi \not\leq t$, for $t \in \{\perp, u, \top\}$

Construction

A **tableau** for a formula φ is constructed as follows:

- ▶ start with $\perp \not\leq \varphi$
- ▶ choose a branch of the tree
- ▶ choose a statement σ on the branch
- ▶ choose a rule with head σ
- ▶ add it at the bottom of the branch
- ▶ repeat until every branch contains one of the following **contradictions**

$$\begin{array}{lll} \perp \not\leq \varphi & s \leq t \text{ with } s \not\leq t & s \leq \varphi \text{ and } t \not\leq \varphi \text{ with } t \leq s \\ \varphi \not\leq \top & s \not\leq t \text{ with } s \leq t & \varphi \leq s \text{ and } \varphi \not\leq t \text{ with } s \leq t \end{array}$$

where $s, t \in \{\perp, u, \top\}$ and φ is a formula

Tableaux Rules

$$\begin{array}{c} t \not\leq \varphi \\ | \\ \varphi \leq s \end{array}$$

$$\begin{array}{c} t \leq \varphi \\ | \\ \varphi \not\leq s \end{array}$$

$$\begin{array}{c} \varphi \not\leq t \\ | \\ s \leq \varphi \end{array}$$

$$\begin{array}{c} \varphi \leq t \\ | \\ s \not\leq \varphi \end{array}$$

s maximal $< t$

$$\begin{array}{c} t \leq \neg\varphi \\ | \\ \varphi \leq \neg t \end{array}$$

$$\begin{array}{c} t \not\leq \neg\varphi \\ | \\ \varphi \not\leq \neg t \end{array}$$

$$\begin{array}{c} t \leq \varphi \wedge \psi \\ | \\ t \leq \varphi \\ | \\ t \leq \psi \end{array}$$

$$\begin{array}{c} t \not\leq \varphi \wedge \psi \\ / \quad \backslash \\ t \not\leq \varphi \quad t \not\leq \psi \end{array}$$

$$\begin{array}{c} \varphi \vee \psi \leq t \\ | \\ \varphi \leq t \\ | \\ \psi \leq t \end{array}$$

$$\begin{array}{c} \varphi \vee \psi \not\leq t \\ / \quad \backslash \\ \varphi \not\leq t \quad \psi \not\leq t \end{array}$$

$t \neq \perp$

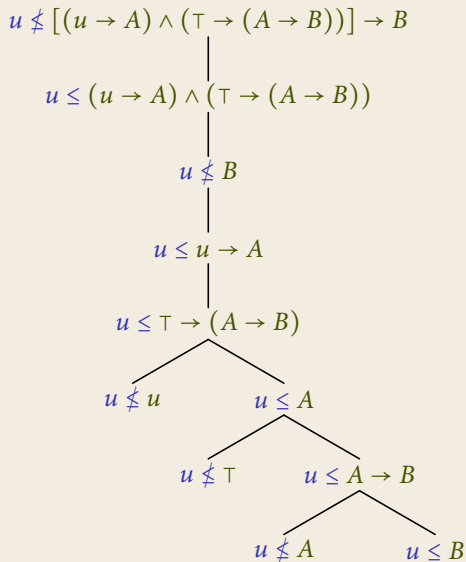
$$\begin{array}{c} \top \not\leq \varphi \rightarrow \psi \\ / \quad \backslash \\ u \leq \varphi \quad \top \leq \varphi \\ | \quad \quad | \\ u \not\leq \psi \quad \top \not\leq \psi \end{array}$$

$$\begin{array}{c} u \not\leq \varphi \rightarrow \psi \\ | \\ u \leq \varphi \\ | \\ u \not\leq \psi \end{array}$$

$$\begin{array}{c} \top \leq \varphi \rightarrow \psi \\ / \quad \backslash \\ \top \not\leq \varphi \quad \top \leq \psi \\ | \\ u \leq \varphi \rightarrow \psi \\ / \quad \backslash \\ u \not\leq \varphi \quad u \leq \psi \end{array}$$

$$\begin{array}{c} \top \leq \varphi \rightarrow \psi \\ / \quad \backslash \\ u \not\leq \varphi \quad u \leq \psi \end{array}$$

Example



Intuitionistic Logic

The constructivists view

- ▶ We are not interested in **truth** but in **provability**.
- ▶ To prove the **existence** of an object is to give a concrete example.

$$\text{prove } \exists x \varphi(x) \quad \Leftrightarrow \quad \text{find } t \text{ with } \varphi(t)$$

- ▶ To prove a **disjunction** is to prove one of the choices.

$$\text{prove } \varphi \vee \psi \quad \Leftrightarrow \quad \text{prove } \varphi \text{ or prove } \psi$$

Goal

A variant of first-order logic that captures these ideas.

Boolean algebras

In **classical logic** the **truth values** form a **boolean algebra** with operations

$$\wedge, \vee, \neg, \top, \perp$$

Properties of negation:

$$x \wedge \neg x = \perp \quad x \vee \neg x = \top$$

Heyting algebras

In **intuitionistic logic** the **truth values** form instead a **Heyting algebra** with operations

$$\wedge, \vee, \rightarrow, \top, \perp$$

Properties of implication:

$$z \leq x \rightarrow y \quad \text{iff} \quad z \wedge x \leq y$$

(that is $x \rightarrow y$ is the largest element satisfying $(x \rightarrow y) \wedge x \leq y$)

$$x \wedge (x \rightarrow y) = x \wedge y$$

$$x \rightarrow x = \top$$

$$y \wedge (x \rightarrow x) = y$$

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z).$$

Heyting algebras

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$$x \rightarrow x = \top$$

$$y \wedge (x \rightarrow x) = y$$

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z).$$

Negation $\neg x := x \rightarrow \perp$

satisfies $x \wedge \neg x = \perp$, but not $x \vee \neg x = \top$

Forcing Frames

Definition

Transition system $\mathfrak{G} = \langle S, \leq, (P_i)_{i \in I}, s_0 \rangle$ with one edge relation \leq that forms a **partial order**:

- ▶ **reflexive** $s \leq s$
- ▶ **transitive** $s \leq t \leq u$ implies $s \leq u$
- ▶ **anti-symmetric** $s \leq t$ and $t \leq s$ implies $s = t$

The forcing relation

\mathfrak{G} forcing frame, $s \in S$ state, φ formula

$s \Vdash P_i$: iff $t \in P_i$ for all $t \geq s$

$s \Vdash \varphi \wedge \psi$: iff $s \Vdash \varphi$ and $s \Vdash \psi$

$s \Vdash \varphi \vee \psi$: iff $s \Vdash \varphi$ or $s \Vdash \psi$

$s \Vdash \neg\varphi$: iff $t \not\Vdash \varphi$ for all $t \geq s$

$s \Vdash \varphi \rightarrow \psi$: iff $t \Vdash \varphi$ implies $t \Vdash \psi$ for all $t \geq s$

The **truth value** of φ in \mathfrak{G} is

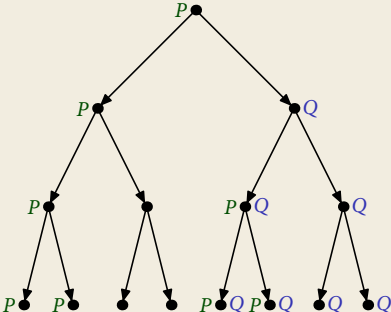
$$\llbracket \varphi \rrbracket_{\mathfrak{G}} := \{ s \in S \mid s \Vdash \varphi \},$$

which is **upwards-closed** with respect to \leq .

Intuition

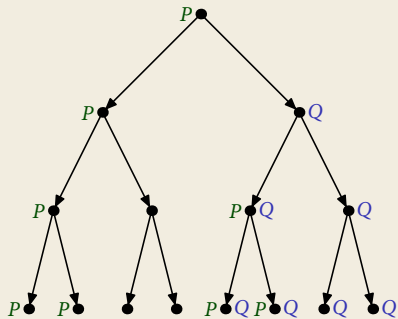
Intuitionistic logic speaks about the **limit behaviour** of φ for large s .

Example



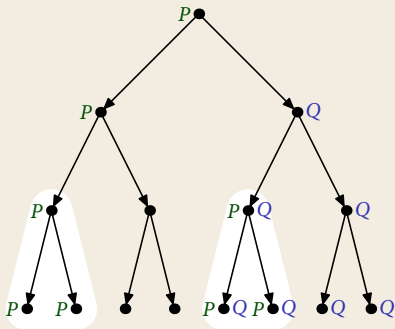
Example

$\varphi := P$



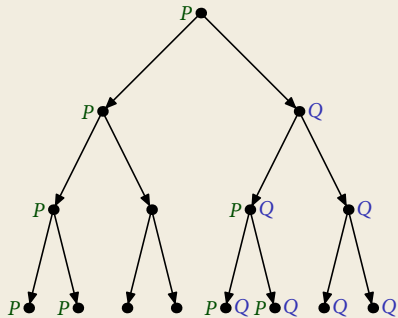
Example

$\varphi := P$



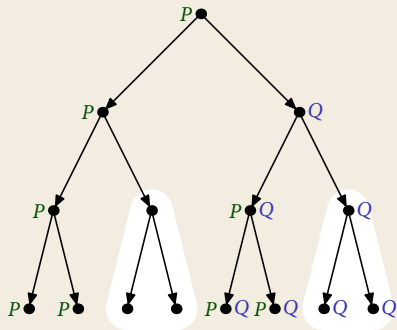
Example

$$\varphi := \neg P$$



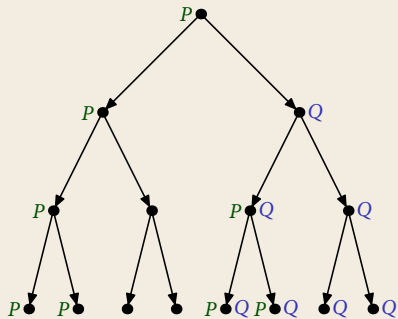
Example

$$\varphi := \neg P$$



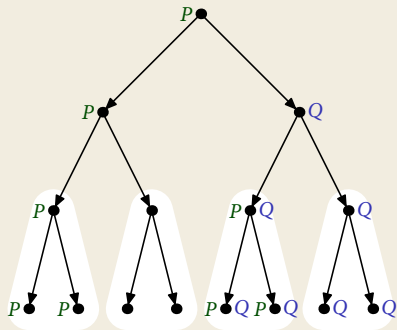
Example

$$\varphi := P \vee \neg P$$



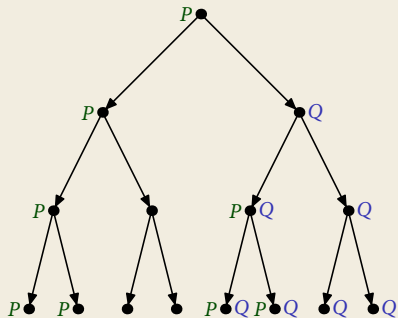
Example

$$\varphi := P \vee \neg P$$



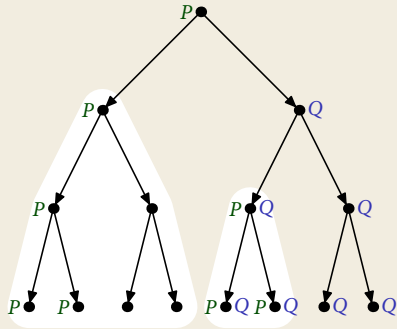
Example

$$\varphi := Q \rightarrow P$$



Example

$$\varphi := Q \rightarrow P$$



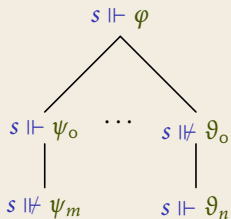
Tableaux for Intuitionistic Logic

Statements

$s \Vdash \varphi$ $s \nVdash \varphi$ $s \leq t$

s, t state labels, φ a formula

Rules



$s \Vdash \varphi$
 $\quad \mid$
 $t \Vdash \varphi$
 φ atomic, $t \geq s$ arbitrary

$s \nVdash \varphi$
 φ atomic

$s \Vdash \neg\varphi$
 $\quad \mid$
 $t \nVdash \varphi$
 $t \geq s$ arbitrary

$s \nVdash \neg\varphi$
 $\quad \mid$
 $s \leq t$
 $\quad \mid$
 $t \Vdash \varphi$
 t new

$s \Vdash \varphi \wedge \psi$
 $\quad \mid$
 $s \Vdash \varphi$
 $\quad \mid$
 $s \Vdash \psi$

 $s \Vdash \varphi \vee \psi$
 $\swarrow \quad \searrow$
 $s \Vdash \varphi \quad s \Vdash \psi$

$s \nVdash \varphi \wedge \psi$
 $\swarrow \quad \searrow$
 $s \nVdash \varphi \quad s \nVdash \psi$

 $s \nVdash \varphi \vee \psi$
 $\quad \mid$
 $s \nVdash \varphi$
 $\quad \mid$
 $s \nVdash \psi$

$s \Vdash \varphi \rightarrow \psi$
 $\swarrow \quad \searrow$
 $t \nVdash \varphi \quad t \Vdash \psi$
 $t \geq s$ arbitrary

$s \nVdash \varphi \rightarrow \psi$
 $\quad \mid$
 $s \leq t$
 $\quad \mid$
 $t \Vdash \varphi$
 $\quad \mid$
 $t \nVdash \psi$
 t new

$s \Vdash \exists x \varphi$
 $\quad \mid$
 $s \Vdash \varphi(c)$
 c new

$s \nVdash \exists x \varphi$
 $\quad \mid$
 $s \nVdash \varphi(c)$
 c arbitrary

$s \Vdash \forall x \varphi$
 $\quad \mid$
 $t \Vdash \varphi(c)$
 c, t arbitrary with $s \leq t$

$s \nVdash \forall x \varphi$
 $\quad \mid$
 $s \leq t$
 $\quad \mid$
 $t \nVdash \varphi(c)$
 c, t new

(' c arbitrary' means either new or appearing somewhere on the same branch.)

$s \Vdash A \rightarrow (B \rightarrow A)$

$s \leq t$

$t \Vdash A$

$t \Vdash B \rightarrow A$

$t \leq u$

$u \Vdash B$

$u \Vdash A$

$u \Vdash A$

$$s \Vdash \exists x(\varphi \vee \psi) \rightarrow (\exists x\varphi \vee \exists x\psi)$$

$$s \leq t$$

$$t \Vdash \exists x(\varphi \vee \psi)$$

$$t \Vdash \exists x\varphi \vee \exists x\psi$$

$$t \Vdash \varphi(c) \vee \psi(c)$$

$$t \Vdash \exists x\varphi$$

$$t \Vdash \exists x\psi$$

$$t \Vdash \varphi(c)$$

$$t \Vdash \psi(c)$$

$$t \Vdash \varphi(c)$$

$$t \Vdash \psi(c)$$

