Exercise 1 We consider (undirected) graphs as structures of the form $\mathfrak{G} = \langle V, E \rangle$ where E is the binary edge relation. Express the following statements in first-order logic.

(a) All vertices are neighbours.

(b) The graph contains a triangle.

(c) Every vertex has exactly three neighbours.

(d) Every pair of vertices is connected by a path of length at most 3.

Exercise 2 Let f be a binary function symbol, g, h unary, and c a constant symbol.

(a) Find the most general unifier for the following pairs of terms.

(i) $f(g(x), y)$ and $f(x, h(y))$

(ii) $f(h(x), x)$ and $f(x, h(y))$

(iii) $f(x, f(x, g(y)))$ and $f(y, f(h(c), x))$

(iv) $f(f(x, c), g(f(y, x)))$ and $f(x, g(x))$

(b) Solve the following set of term equations

$$x = f(y, z), \quad y = g(u), \quad z = h(y), \quad u = f(v, w), \quad v = f(c, w).$$

Exercise 3 Consider the following formulae.

(a) $\exists x \exists y \forall z [z = x \vee z = y]$

(b) $\forall x [\exists y R(x, y) \rightarrow \exists y R(y, x)]$

(c) $\forall x [\forall y \exists z [R(x, f(y, z))] \rightarrow \forall y \forall z [R(f(x, y), f(x, z)) \vee R(y, z)]]$

(d) $\exists x \forall y R(x, y) \wedge \forall x \exists y R(x, y) \wedge \forall x \forall y [R(x, y) \rightarrow \exists z [R(x, z) \wedge R(z, x)]]$

For each of them

(1) transform it into Skolem normal form;

(2) transform it into a set of clauses.

Exercise 4 Use the resolution method to check that the following formulae are inconsistent.

(a) $\forall x \forall y [x \leq y \rightarrow (P(x) \leftrightarrow P(y))] \wedge \forall x \forall y [x \leq y \vee y \leq x] \wedge \exists x P(x) \wedge \exists x \neg P(x)$

(b) $\forall x \exists y [y \leq x \wedge \neg E(x, y)] \wedge \forall x \forall y [x \leq y \wedge y \leq x \rightarrow E(x, y)] \wedge \exists x \forall y [x \leq y]$

(c) $\forall x \forall y [R(x, y) \rightarrow (P(x) \leftrightarrow \neg P(y))] \wedge \forall x \forall y [R(x, y) \rightarrow \exists z [R(x, z) \wedge R(z, y)]] \wedge \exists x \exists y R(x, y)$

(d) $\forall x R(x, f(x)) \wedge \forall x \forall y \forall z [R(x, y) \wedge R(y, z) \rightarrow R(x, z)] \wedge \forall x \forall y [E(x, y) \rightarrow \neg R(x, y)]$
$\wedge \exists x E(x, f(f(x)))$

Exercise 5 Use SLD resolution to check that the following set of Horn-formulae is inconsistent.

(a) $\forall x T(x,x)$,
$\forall x \forall y \forall z [E(x,y) \wedge T(y,z) \to T(x,z)]$,
$E(a,b)$,
$E(b,c)$,
$E(c,d)$,
$\neg T(a,d)$.

(b) $\forall x T(x,x)$,
$\forall x \forall y \forall z [T(x,y) \wedge E(y,z) \to T(x,z)]$,
$E(a,b)$,
$E(b,c)$,
$E(c,d)$,
$\neg T(a,d)$.

(c) $R(c,x,x)$,
$\forall x \forall y \forall z \forall w [R(x, f(y,z), w) \to R(f(y,x), z, w)]$,
$\neg \forall x \forall y [R(f(x, f(y,c)),\ c,\ f(y, f(x,c)))]$.