Exercise 1 Show that the following formulae are valid using tableau proofs.

(a) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$

(b) $\psi \rightarrow ((\varphi \wedge \psi) \vee \psi)$

(c) $(\neg \psi \rightarrow \neg \varphi) \rightarrow (\varphi \rightarrow \psi)$

(d) $\varphi \rightarrow \neg\neg \varphi$

(e) $((\varphi \wedge \psi) \vee \psi) \rightarrow \psi$

(f) $(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \vartheta) \rightarrow (\varphi \rightarrow \psi \wedge \vartheta)$

(g) $(\varphi \rightarrow \psi \wedge \vartheta) \rightarrow (\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \vartheta)$

(h) $\neg\neg \varphi \rightarrow \varphi$

(i) $\varphi \vee \neg \varphi$

(j) $\neg(\neg \varphi \wedge \neg \psi) \rightarrow (\varphi \vee \psi)$

(k) $\varphi \rightarrow \exists x \varphi$

(l) $\forall x \varphi \rightarrow \varphi$

(m) $\forall x R(x, x) \rightarrow \forall x \exists y R(f(x), y)$

(n) $\exists x (\varphi \vee \psi) \rightarrow (\exists x \varphi \vee \exists x \psi)$

(o) $(\exists x \varphi \vee \exists x \psi) \rightarrow \exists x (\varphi \vee \psi)$

(p) $\forall x \varphi \wedge \forall x \psi \rightarrow \forall x (\varphi \wedge \psi)$

(q) $\forall x (\varphi \wedge \psi) \rightarrow \forall x \varphi \wedge \forall x \psi$

(r) $\forall x \forall y [\varphi(x) \leftrightarrow \varphi(y)] \wedge \exists x \varphi(x) \rightarrow \forall x \varphi(x)$

Exercise 2 Prove that the formulae from Exercise 1 are valid using Natural Deduction.

Exercise 3 Find all consistent sets for the following sets of rules.

(a) $\quad \dfrac{}{\alpha} \quad \dfrac{\alpha : \beta}{\delta} \quad \dfrac{\alpha : \gamma}{\delta}$

(b) $\quad \dfrac{}{\alpha} \quad \dfrac{\alpha : \beta \, \gamma}{\beta}$

(c) $\quad \dfrac{}{\alpha} \quad \dfrac{\alpha \, \beta}{\gamma} \quad \dfrac{\alpha : \gamma}{\beta}$

Exercise 4 For each of the following subsets $\Phi \subseteq \mathcal{P}(\{\alpha, \beta\})$, find a set of rules R such that Φ is the set of all consistent sets for R.

(a) $\{\varnothing, \{\alpha\}, \{\alpha, \beta\}\}$

(b) $\{\{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$

(c) $\{\varnothing, \{\alpha, \beta\}\}$

(d) $\{\{\alpha\}, \{\alpha, \beta\}\}$

Exercise 5 Derive the following additional rules from the basic ones of the Natural Deduction calculus.

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi}{\Gamma, \Delta \vdash \varphi} \qquad \frac{\Gamma, \neg\varphi \vdash \neg\psi}{\Gamma \vdash \psi \to \varphi} \qquad \frac{\Gamma, \neg\varphi \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

Exercise 6 Find a rule for proofs by induction.

$$\frac{\cdots \quad \cdots}{\forall x \varphi(x)} \qquad \text{where } \varphi(x) \text{ is a formula talking about natural numbers.}$$