# IA168 — Problem set 4

## Problem 1 [4 points]

Consider incomplete-information game $G = (\{1,2\}, (\{A,B,C\}, \{D,E,F\}), (\{P,Q\}, \{R,S\}), (u_1,u_2)\})$, where $u_1, u_2$ are given by the following matrices:

| $u_1(-,-,P)$ | $D$ | $E$ | $F$ |
|---|---|---|---|
| $A$ | 6 | 5 | 4 |
| $B$ | 1 | 2 | 5 |
| $C$ | 1 | 2 | 3 |

| $u_1(-,-,Q)$ | $D$ | $E$ | $F$ |
|---|---|---|---|
| $A$ | 6 | 5 | 4 |
| $B$ | 1 | 2 | 3 |
| $C$ | 1 | 5 | 3 |

| $u_2(-,-,R)$ | $D$ | $E$ | $F$ |
|---|---|---|---|
| $A$ | 6 | 1 | 1 |
| $B$ | 5 | 1 | 1 |
| $C$ | 4 | 1 | 2 |

| $u_2(-,-,S)$ | $D$ | $E$ | $F$ |
|---|---|---|---|
| $A$ | 1 | 5 | 1 |
| $B$ | 2 | 4 | 2 |
| $C$ | 3 | 3 | 3 |

For each $X \in \{A,B,C,D,E,F\}$, find all strictly, weakly, and very weakly dominant strategies in game $G_{-X}$, where $G_{-X}$ is created from $G$ by deleting action $X$.

## Problem 2 [8 points]

Consider "3$^{\text{rd}}$ price auction" as a game of incomplete information. The payoff of every player is 0 if their bid was not (strictly) highest, and it is their type minus the 3$^{\text{rd}}$ highest bid if they were the highest bidder. The bid is a non-negative real number.

Formally, consider the following game of incomplete information

$$G = (N, (A_i)_{i\in N}, (T_i)_{i\in N}, (u_i)_{i\in N}),$$

where $N = \{1,2,3,\dots,n\}$, $n \geq 3$, $(\forall i \in N)\ A_i = T_i = \mathbb{R}_0^+ = \{r \in \mathbb{R} \mid r \geq 0\}$, and

$$u_i(a_1,\dots,a_n;t_i) = \begin{cases} 0 & (\exists j \in N)\ a_j \geq a_i, \\ t_i - a_{i_3} & a_{i_1} > a_{i_2} \geq a_{i_3} \geq \dots \geq a_{i_n}, i_1 = i, \{i_1,\dots,i_n\} = \{1,\dots,n\}. \end{cases}$$

a) Prove that there is no ex-post Nash equilibrium.

b) Prove or disprove the existence of an ex-post Nash equilibrium if the bids of each player are bounded by a common bound, i.e.,

$$(\exists v_{max} \in \mathbb{R}_0^+)\ (\forall i \in N)\ A_i = [0, v_{max}].$$

c) Prove or disprove the existence of an ex-post Nash equilibrium if the types of each player are bounded by a common bound, i.e.,

$$(\exists v_{max} \in \mathbb{R}_0^+)\ (\forall i \in N)\ T_i = [0, v_{max}].$$

d) Prove or disprove the existence of an ex-post Nash equilibrium if the bids of each player are bounded by possibly different bounds, i.e.,

$$(\exists v_1,\dots,v_n \in \mathbb{R}_0^+)\ (\forall i \in N)\ A_i = [0, v_i].$$

## Problem 3 [8 points]

Consider the following Bayesian game: There are two players, they have two actions $A, B$, and they have two types $S, R$. Type $S$ means the player wants to play the same action as the other player, $R$ means he wants to play the other action. Specifically, the gain is $+3$ if this goal is achieved, plus there is bonus $+1$ for playing action $A$.

Formally: $G_P = (\{1,2\}, (\{A,B\},\{A,B\}), (\{S,R\},\{S,R\}), (u_1,u_2), P)$, where $u_1, u_2$ are given by the following matrices:

| $u_1(-,-,S)$ | $A$ | $B$ |
|---|---|---|
| $A$ | 4 | 1 |
| $B$ | 0 | 3 |

| $u_1(-,-,R)$ | $A$ | $B$ |
|---|---|---|
| $A$ | 1 | 4 |
| $B$ | 3 | 0 |

| $u_2(-,-,S)$ | $A$ | $B$ |
|---|---|---|
| $A$ | 4 | 0 |
| $B$ | 1 | 3 |

| $u_2(-,-,R)$ | $A$ | $B$ |
|---|---|---|
| $A$ | 1 | 3 |
| $B$ | 4 | 0 |

Let $\mathrm{BNE}(G_P)$ denote the set of Bayesian Nash equilibria in game $G_P$. Moreover, let $UV|XY$ denote the strategy profile $(\{(S,U),(R,V)\},\{(S,X),(R,Y)\})$ (i.e., player 1 plays $U$ if he is $S$ and he plays $V$ if he is $R$; similarly for player 2). Find a distribution $P$ such that:

a) $\mathrm{BNE}(G_P) = \emptyset$;

b) $\mathrm{BNE}(G_P) = \{AA|AB, AB|AA\}$;

c) $\mathrm{BNE}(G_P) = \{AB|AB\}$;

d) $\mathrm{BNE}(G_P) = \{AB|AB, BA|BA\}$;

e) $\mathrm{BNE}(G_P) = \{AA|AB\}$;

f) $|\mathrm{BNE}(G_P)| = 5$.

We further require that $P$ satisfies that for every player $i \in \{1,2\}$ and every type $t \in \{S,R\}$, the probability that $i$ is of type $t$ is positive.