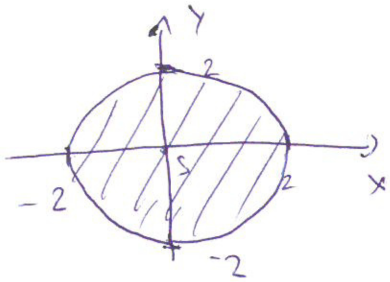
Pr 1 $x^2+y^2 \le 4 \quad \wedge \quad y \ge \frac{1}{x}$

$\boxed{\text{Cvičko 1}}$

1

kreslíme hranicu $x^2+y^2=2^2$, $y=\frac{1}{x}$

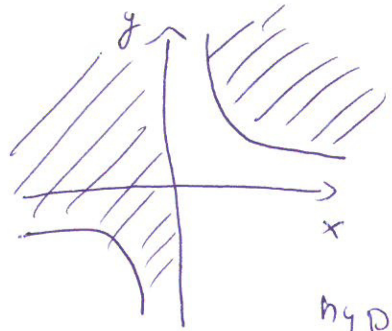


dosadíme napr. $S=[0,0]$

a vidíme $0 \le 4$

$\Rightarrow$ vnútorná oblasť

spĺňa $\le$



hyperbola

overíme napr.

$[1,1]$ spĺňa rovnosť

$[1,2]$ $\frac{1}{1} \le 2$ ✓

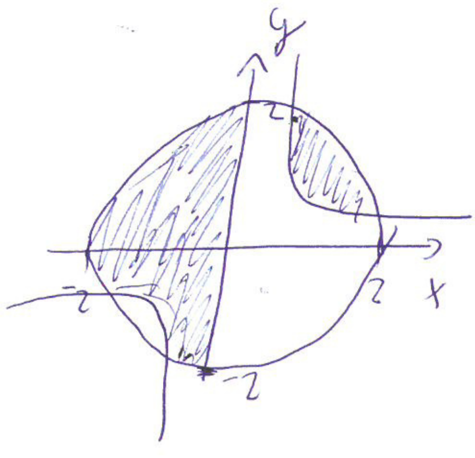
$[1,-2]$ $-1 \not\le -2$

$[-1,-1]$ spĺňa rovnosť

a čo vieme povedať o $x=0$?

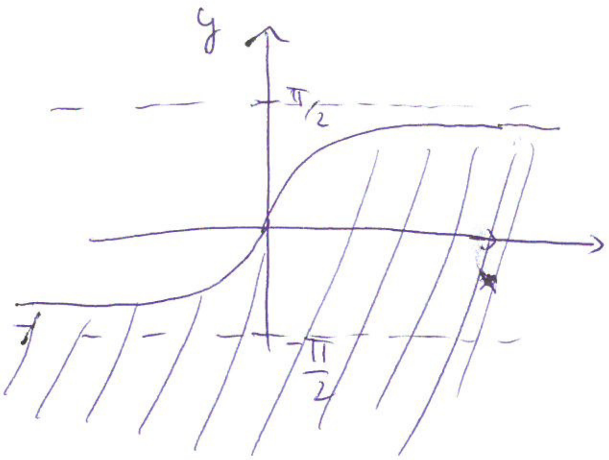
$\lim_{x\to 0^-} \frac{1}{x} = -\infty$ teda $-\infty \le$ ale to leží

Spolu



1b) $y \leq \arctan(x)$, $y \leq \frac{1}{x^2}$

$y = \arctan(x)$

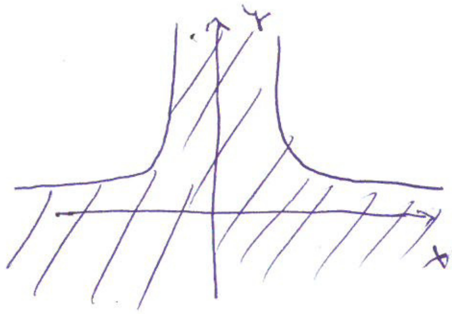


napr bod $[0,0]$ splňuje $y = \arctan(x)$

a $[0, -\pi/2]$ splňuje nerovnosť

$-\pi/2 \leq \arctan(0) = 0$

$y = \frac{1}{x^2}$



napr. $[1,1]$ splňuje $y = \frac{1}{x^2}$

$[1,-2]$ splňuje $-2 \leq \frac{1}{1} = 1$

a zrejme $\lim_{x \to 0} \frac{1}{x^2} = \infty$,

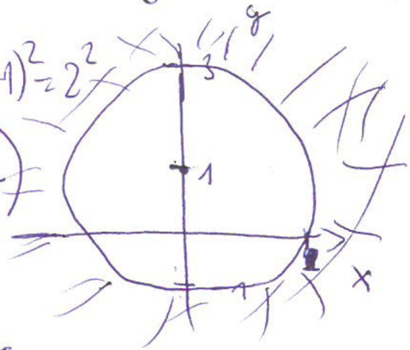
preto $x = 0$ splňuje nerovnosť

riešenie:

c) $x^2+(y-1)^2 \geq 4$, $y+x^2-2x \geq 0$, $y \geq 0$

$x^2+(y-1)^2=2^2$

$k([0,1],2)$

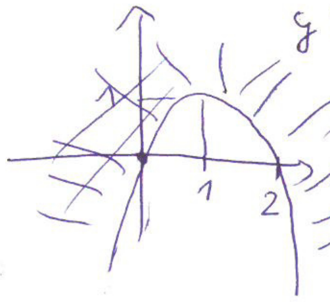


napr. $[0,0]$ nesplňuje $x^2+(y-1)^2 \geq 4$

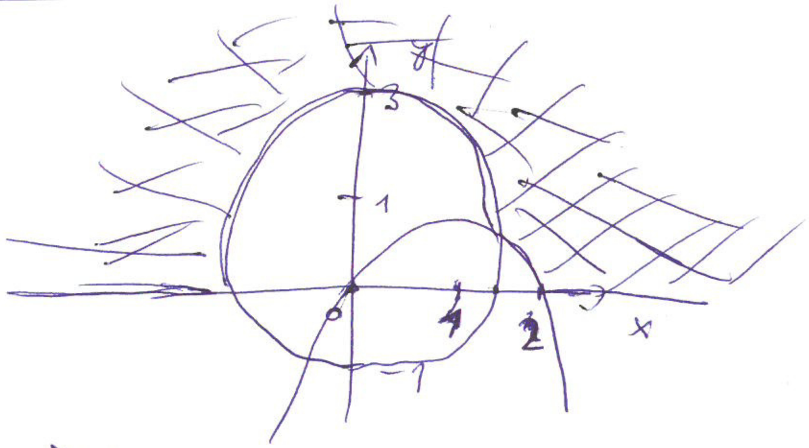
„obrátená parabola"

$y = 2x - x^2 = x(2-x)$

$y \geq x(2-x)$



napr. $[1,0]$ nesplňuje nerovnosť



Pr2 a) $\dfrac{x \cdot y}{y(x^3+x^2+x+1)}$

$y \cdot (x^3+x^2+x+1) \neq 0$

$\Rightarrow y \neq 0 \wedge (x^3+x^2+x+1) \neq 0$

$(x^3+x^2+x+1) = x^2(x+1) + 1\cdot(x+1) = (x+1)(x^2+1) \neq 0 \Leftrightarrow \underline{\underline{x \neq -1}}$

alebo $(x^4-1) = (x-1)(x^3+x^2+x+1)$

má korene nad $\mathbb{C}$ 4-té odmocniny z 1, z toho reálne sú iba ± 1, preto $x \neq -1$ ∇ (jednoznačnosť rozkladu polynómov)

$D_f = \{(x,y) \in \mathbb{R}^2 \mid x \neq -1, y \neq 0\}$

c) $\ln(-x^2-y^2)$

$g(x) = \ln(x)$ má $D_g = \{x \in \mathbb{R} \mid x > 0\}$

a vieme, že $-(x^2+y^2) \leq 0$, preto

$D_f = \emptyset$.

b) $\ln(x^2-y^2)$ ④

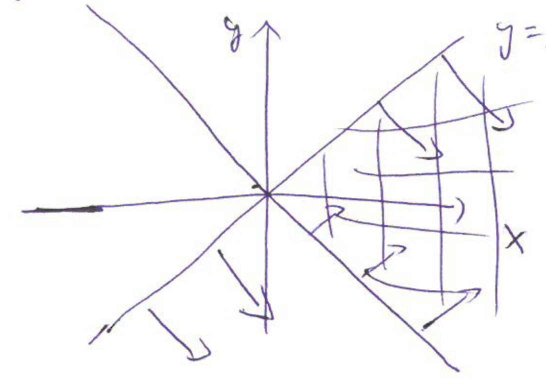
$x^2-y^2>0$ $\qquad (x-y)(x+y)>0$

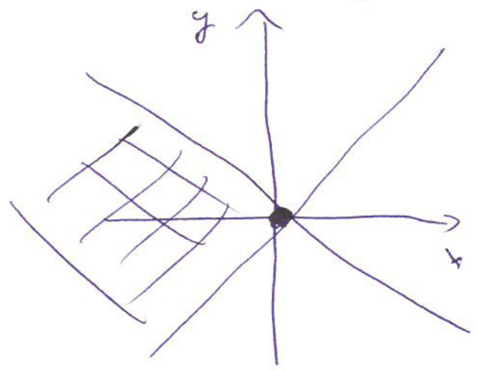
$(x-y>0 \wedge x+y>0) \quad (*)$

$\vee$

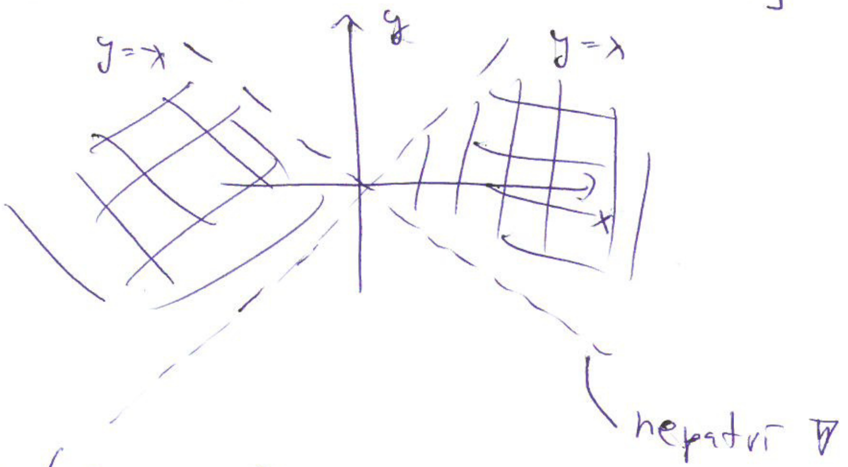
$(x-y<0 \wedge x+y<0) \quad (**)$

$y=-x \qquad (*) \quad x>y,\ y>-x$ $\qquad (**)$ - symetricky





$$D_f=\{(x,y)\in\mathbb{R}^2 \mid (*) \vee (**)\}$$



nepatrí ∇

d) $\arcsin(2\chi_Q(x))$

$$\chi_Q(x)=\begin{cases}0 & x\in\mathbb{R}-\mathbb{Q}=\mathbb{I}\\ 1 & x\in\mathbb{Q}\end{cases}$$

racionálne číslo

$g(x)=\arcsin(x)$

$D_g=[-1,1]$

$H_{\chi_Q}=\{0,1\} \Rightarrow H_{2\cdot\chi_Q}=\{0,2\}$

obor hodnôt χ_Q

$H_{2\cdot\chi_Q}\cap D_g=\{0\}$, preto

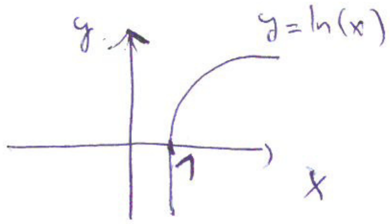
$$D_f=\chi_Q^{-1}(\{0\})$$

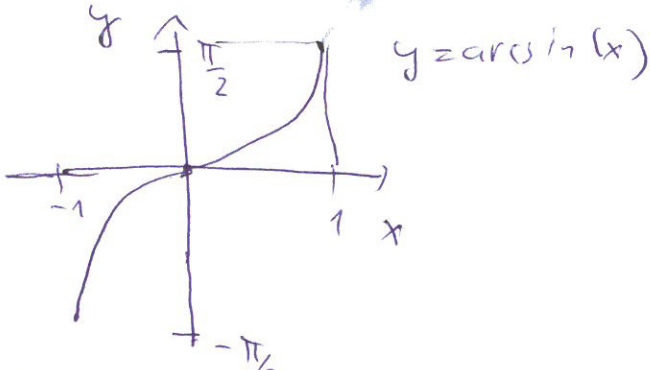
$$=\underset{x}{\mathbb{I}}\times\underset{y}{\mathbb{R}}$$

c) $f(x,y,z) = \sqrt{\ln(x) \cdot \arcsin(y^2 z)}$ ⑤

$$\ln(x) \cdot \arcsin(y^2 z) \geq 0 \quad \wedge \quad x > 0 \quad \wedge \quad y^2 z \in [-1,1]$$

$$(\ln(x) \geq 0 \wedge \arcsin(y^2 z) \geq 0) \vee (\ln(x) \leq 0 \wedge \arcsin(y^2 z) \leq 0)$$





$$(x \in [1,\infty) \wedge y^2 z \geq 0) \vee (x \in (0,1\rangle \wedge y^2 z \leq 0)$$

$$(*) \quad (x \in [1,\infty) \wedge z \geq 0) \vee (x \in (0,1) \wedge z \leq 0)$$

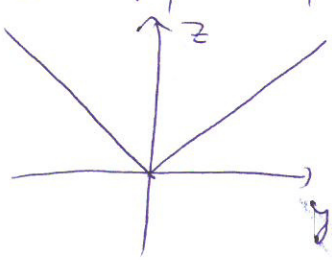
$$(**) \quad y^2 z \in [-1,1]$$

$$D_f = \{ (x,y,z) \in \mathbb{R}^3 \mid (*) \text{ a } (**) \text{ platí} \}$$

Pr 3 a) $f(x,y) = \sqrt{x^2 + y^2}$

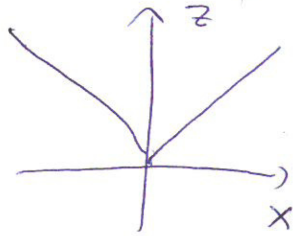
Rez rovinou $x = 0$

$z = \sqrt{y^2} = |y|$



Rez rovinou $y = 0$

$z = \sqrt{x^2} = |x|$

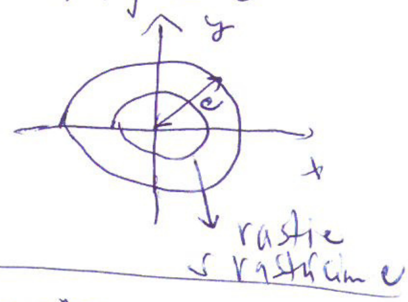


VRSTEVNICE pre c:

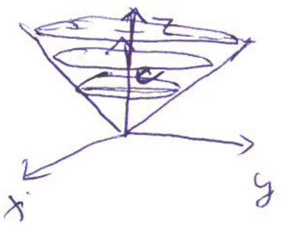
$\sqrt{x^2+y^2} = c$

$\Rightarrow \boxed{c \geq 0}$

$x^2 + y^2 = c^2$



$\Rightarrow$ graf je kužel s vrcholom v $[0,0,0]$

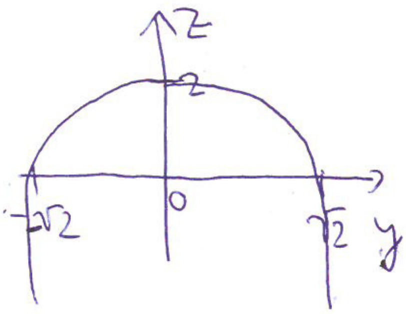


b) $f(x,y) = 2 - x^2 - y^2$

rez rovinou

$x = 0$

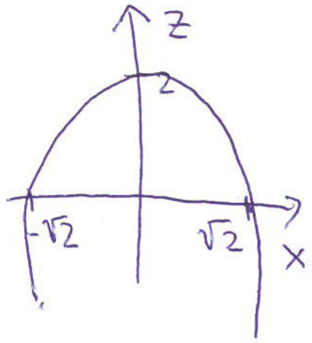
$z = 2 - y^2 = (\sqrt{2} - y)(\sqrt{2} + y)$



rez rovinou

$y = 0$

$z = 2 - x^2$



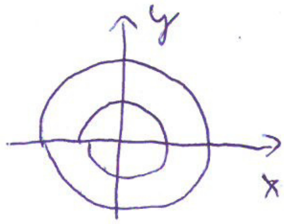
vrstevnice v c

$2 - x^2 - y^2 = c$

$2 - c = x^2 + y^2$

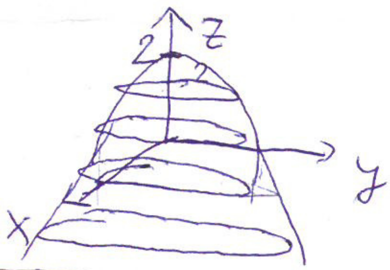
$\Rightarrow 2 - c \geq 0$

$c \geq 2$



polomer $\sqrt{2-c}$ – s klesajúcim c rastie.

Výsledok: obrátený paraboloid s vrcholom v bode $(0,0,2)$

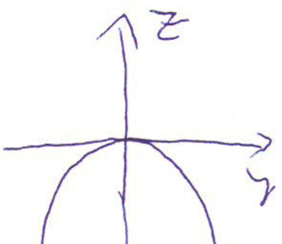


c) $f(x,y) = x^2 - y^2$ (pre celkový graf použite napr. WOLFRAM ALPHA)

rez rovinou

$x = 0$

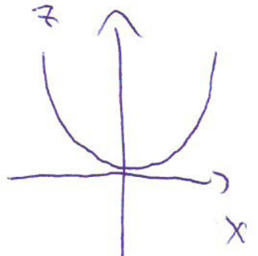
$z = -y^2$



rez rovinou

$y = 0$

$z = x^2$



vrstevnice

$x^2 - y^2 = c$

$x^2 - c = y^2 \Rightarrow x^2 - c \geq 0$

$y = \pm\sqrt{x^2 - c}$ $\boxed{\text{podobné ako c)}}$

$\boxed{k = -c} > 0$ $x^2 = y^2 + c$

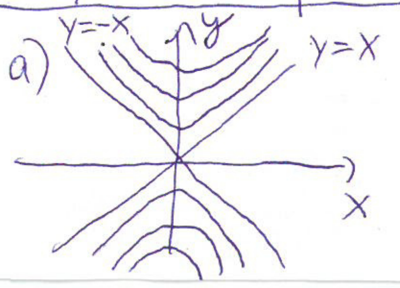
UVAŽUJME a) • $c \leq 0$ $x = \pm\sqrt{y^2 - k}$ $\underline{y^2 \geq k}$

b) • $c = 0$ $y = \pm|x|$

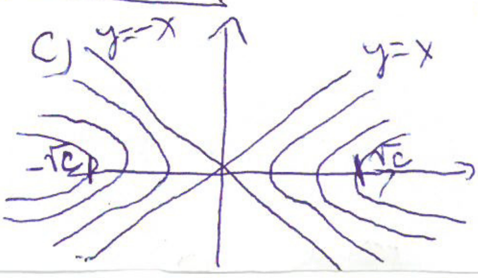
c) • $c > 0 \wedge x^2 \geq c$

↳ s klesajúcim c rastie

a)

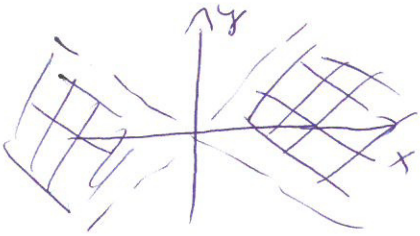


c)



d) $f_d(x,y) = \ln(x^2-y^2)$

Porovnanie s c) → iný definičný obor viď 2b) !



jediný rozdiel je „preškálovanie logaritmom."

t.j.

lebo $f_d(x,y) = \ln(f(x,y))$ ← z c)

rez rovinou $x=0$

$z = \ln(-y^2)$

$\emptyset$

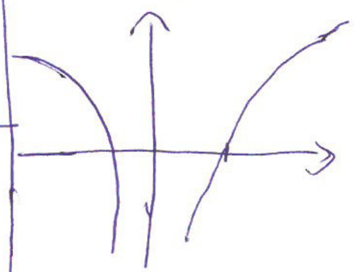
$y = \pm x$

nemôže dosiahnuť

$\emptyset$

rez rovinou $y=0$

$z = \ln(x^2)$



vrstevnice

$\ln(x^2-y^2) = k$

$x^2-y^2 = e^k$

vieme, že $e^k > 0$, teda postupuje úplne rovnako ako v c), keď položíme $c = e^k > 0$.

Limity

$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y)}$

Návody:

1. dosadiť (x_0,y_0) a zistiť typ limity, používame $\frac{1}{\infty} = 0$, $\frac{1}{0_+} = +\infty$, $\frac{1}{0_-} = -\infty$

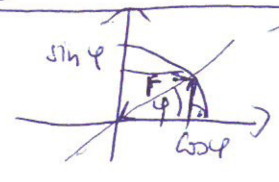
NEURČITÉ VÝRAZY $0\cdot\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, ∞^0, 0^0, 1^∞

– NEMÔŽEME POUŽIŤ L'Hospitalovo pravidlo! v prípade funkcie 2 a viac premenných

2. substitúcia napr. $t = x\cdot y$

3. transformácia do polárnych súradníc

$$\boxed{\begin{array}{ll} x = r\cos\varphi & \varphi\in[0,2\pi] \\ y = r\sin\varphi & r \geq 0 \end{array}}$$



typicky $f(x,y) = x^2+y^2$, $g(x,y) = x^2-y^2$ atď.

4. „lim 0 · ohraničený výraz = 0"

↓ napr. $\sin(x)$, $\cos(g(x))$

5 NEEXISTENCIA LIMITY

VYUŽÍVANIE RÔZNYCH VZŤAHOV MEDZI PREMENNÝMI:

$x=0,\ y=0,\ y=k\cdot x,\ y=k\cdot x^2$ atď.

$y=e^x$

Pr 4

a) $$\lim_{(x,y)\to(e^2,1)} \frac{\ln x}{y} = \frac{\ln(e^2)}{1} = 2$$ (dosadenie)

[$f(x,y)$ je spojitá v bode $(e^2,1)$]

b) $$\lim_{(x,y)\to(4,4)} \frac{\sqrt{x}-\sqrt{y}}{x-y} = \lim_{(x,y)\to(4,4)} \frac{\cancel{\sqrt{x}-\sqrt{y}}}{(\cancel{\sqrt{x}-\sqrt{y}})(\sqrt{x}+\sqrt{y})} = \frac{1}{4}$$

vieme $\frac{\sqrt{x}-\sqrt{y}}{x-y}$ $\boxed{\text{spojito dodefinovať}}$ v bode $(4,4)$ a to hodnotou $\frac{1}{4}$.

c) $$\lim_{(x,y)\to(1,\infty)} \frac{\cos(y)}{x+y} = \lim_{(x,y)\to(1,\infty)} \frac{1}{x+y}\cdot\cos(y) = 0$$

$\frac{1}{x+y} \to \frac{1}{\infty} = 0$, $\quad |\cos(y)| \le 1$

d) $$\lim_{(x,y)\to(0,2)} \frac{e^{xy}-1}{x\cdot y}\cdot y$$ Poznáme $$\lim_{x\to 0} \frac{e^x-e^0}{x-0} = 1 = (e^x)'\Big|_{x=0}$$

Preto $t = x\cdot y \to 0\cdot 2 = 0$

súčin, ak má zmysel

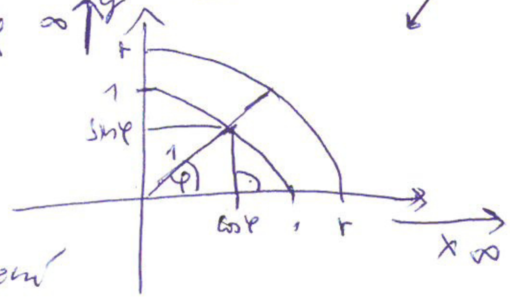
$$\lim_{(t,y)\to(0,2)} \frac{e^t-1}{t}\cdot y = \left(\lim_{t\to 0}\frac{e^t-1}{t}\right)\cdot\left(\lim_{y\to 2} y\right) = 2$$

e) $$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2+y^2}{x^4+y^4}$$ POL. SÚRADNICE

$x = r\cos\varphi$, $\quad \varphi\in[0,\frac{\pi}{2}] \to$ 1. kvadrant

$y = r\cdot\sin\varphi$

$r>0$



$$= \lim_{r\to\infty} \frac{r^2}{r^4(\cos^4\varphi+\sin^4\varphi)}$$ → chceme, aby bolo obmedzené

$$= \lim_{r\to\infty} \frac{1}{r^2}\cdot \boxed{\frac{1}{\cos^4\varphi+\sin^4\varphi}}$$

Pre $r \to \infty$ je $\frac{1}{r^2} \to 0$

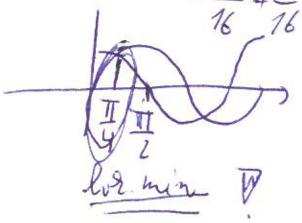
$0 \leq \frac{1}{\cos^4\varphi + \sin^4\varphi} \overset{?}{\leq} H$ ekvivalentne $g(\varphi) = \cos^4\varphi + \sin^4\varphi \geq M > 0$

Chceme nájsť M také, že

Podľa vety z prednášky (Weierstrassova veta) spojitá funkcia $g(\varphi)$ nadobúda na uzavretom intervale $[0, \pi/2]$ svoje minimum, ktoré je > 0, pretože $\cos^4\varphi + \sin^4\varphi \neq 0 \; \forall \varphi$.

$\cos^4(\frac{\pi}{4}) + \sin^4(\frac{\pi}{4}) = \frac{4}{16} + \frac{4}{16} = \frac{1}{2}$

Preto
$$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2+y^2}{x^4+y^4} = 0 .$$



f) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x+y} \left|\frac{0}{0}\right|$ máme kvadráty premenných

skúsme $x = r\cdot\cos\varphi$
$y = r\cdot\sin\varphi$

$$= \lim_{r\to 0_+} \frac{r^2}{r(\cos\varphi+\sin\varphi)} = \lim_{r\to 0_+} r\cdot\frac{1}{\cos\varphi+\sin\varphi}$$

nemusí byť obmedzená

~~vyskúšame~~ zvolíme $y = x$, čo je správne, lebo ak $x \to 0$, potom $y = x \to 0$

$$\lim_{(x,x)\to(0,0)} \frac{2x^2}{2x} = \lim_{x\to 0} x = \boxed{0}$$

zvolíme $y = 1-e^x$, pre $x \to 0$ máme $y \to 1-e^0 = 0$

$$\lim_{x\to 0} \frac{x^2+(1-e^x)^2}{x+(1-e^x)} = \left|\frac{0}{0}\right| \overset{L'H.P}{=} \lim_{x\to 0} \frac{2x+2(1-e^x)\cdot(-e^x)}{1-e^x} \left|\frac{0}{0}\right| L'H.P$$

$$= \lim_{x\to 0} \frac{2 + \overbrace{2(1-e^x)(-e^x)}^{=0} + 2\cdot e^{2x}}{-e^x} = \boxed{-4}$$

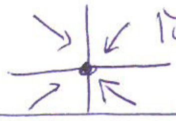
AVŠAK $-4 \neq 0$, preto pôvodná limita neexistuje.

g) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

kvadráty → skúsme polárne súradnice

$x = r\cos\varphi$ $\quad r \to 0_+,\ \varphi \in [0, 2\pi]$

$y = r\sin\varphi$



ľubovoľný uhol

$= \lim\limits_{r\to 0_+} \dfrac{\cos^2\varphi - \sin^2\varphi}{1}$ ↘ ohraničená!

$= \cos(2\varphi)$ → vidíme, že hodnota závisí na uhle φ, preto lim. neexistuje.

Iný prístup: zvolíme $x = 0$

$\lim\limits_{y\to 0} \dfrac{-y^2}{y^2} = -1$

Podobne $y = 0$

$\lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$

$\Rightarrow$ limita neexistuje.

(využili sme, že x^2-y^2 nie je symetrické)

h) $\lim\limits_{(x,y)\to(1,-2)} \dfrac{2x+xy-y-2}{x^2+y^2-2x+4y+5}$ UPRAVÍTE NA SÚČET KVADRÁTOV

$(*)\ \dfrac{x(2+y)-(y+2)}{(x-1)^2-1+(y+2)^2-4+5} = \dfrac{(y+2)(x-1)}{(x-1)^2+(y+2)^2}$ →

$k([1,-2], r)$

kvadráty, môžeme skúsiť posunuté polárne súradnice:

$x = r\cdot\cos\varphi + 1$

$y = r\cdot\sin\varphi - 2$

voľme tak, aby sme sa zbavili konštanty! t.j. stred kružnice bude v počiatku!

ak $(x,y)\to(1,-2)$, potom $r \to 0_+$, $\varphi \in [0, 2\pi]$

$= \dfrac{r^2 \cancel{\cos} \sin\varphi\cdot\cos\varphi}{r^2} = \dfrac{\sin(2\varphi)}{2}$ t.j. ohraničená!

$\lim\limits_{r\to 0_+} \dfrac{\sin(2\varphi)}{2} = \dfrac{\sin(2\varphi)}{2} \Rightarrow$ NEEXISTUJE

alebo zvolíme $y = k\cdot(x-1)-2$

ak $x \to 1$, potom $k\cdot(x-1)-2 \to -2$ správne

$(*) = \dfrac{k(x-1)^2}{(k^2+1)(x-1)^2} = \dfrac{k}{(k^2+1)}$

$\lim\limits_{(x,y)\to(1,-2)} (*) = \dfrac{k}{k^2+1}$ NEEXISTUJE

Pr 5 b) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & [x,y]\neq[0,0] \\ \boxed{0} & \text{inak} \end{cases}$

Ukážte, že f je spojitá na $\mathbb{R}^2$

Zrejme f je spojitá pre $[x,y]\neq[0,0]$ (pri výpočte limity môžeme dosadiť vždy $[x_0,y_0]$)

Zostáva overiť, že

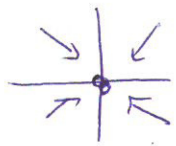
$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0) \quad \text{(definícia spojitosti)}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

kvadráty, použijeme polárne súradnice $x = r\cdot\cos\varphi$, $y = r\cdot\sin\varphi$

$\varphi\in[0,2\pi]$

$r\to 0_+$



$$\lim_{r\to 0_+} \frac{r^3\cdot(\cos^3\varphi+\sin^3\varphi)}{r^2} = \left(\lim_{r\to 0_+} r\cdot \underbrace{(\cos^3\varphi+\sin^3\varphi)}_{\text{je ohraničená, lebo } |\cos\varphi|, |\sin\varphi|\leq 1.}\right) = 0$$

t.j. f je spojitá na $\mathbb{R}^2$.