TAYLOROV POLYNÓM STUPNA 2 FUNKCIE $f: \mathbb{R}^2 \to \mathbb{R}$ v bode (x_0, y_0)

je $$\underset{(x_0,y_0)}{Tf}(x,y) = f(x_0,y_0) + \underset{(x-x_0,\,y-y_0)}{D^1 f(x_0,y_0)} + \underset{(x-x_0,\,y-y_0)}{D^2 f(x_0,y_0)} \cdot \frac{1}{2!}$$

$$\underset{(x-x_0,\,y-y_0)}{D^1 f(x_0,y_0)} = f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0) = \left(f'_x(x_0,y_0),\ f'_y(x_0,y_0)\right)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$$\underset{(x-x_0,\,y-y_0)}{D^2 f(x_0,y_0)} = (x-x_0,\ y-y_0)\underbrace{\begin{pmatrix} f''_{xx}(x_0,y_0) & f''_{xy}(x_0,y_0) \\ f''_{yx}(x_0,y_0) & f''_{yy}(x_0,y_0) \end{pmatrix}}_{Hf(x_0,y_0)}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

Hessova matica

ak má f spojité derivácie derivácie f''_{yx}, f''_{xy} na okolí (x_0,y_0),
potom
$$f''_{yx}(x,y) = f''_{xy}(x,y) \text{ na okolí } (x_0,y_0)$$
(Schwarzova veta) t.j. matica $Hf(x_0,y_0)$ je symetrická.

Pr1 $f(x,y) = x^4 y + xy^2 + x + 2$ v bode $[1,1]$

$f'_x = 4x^3 y + y^2 + 1 \overset{(1,1)}{\rightsquigarrow} 6$

$f'_y = x^4 + 2xy \overset{(1,1)}{\rightsquigarrow} 3$

$f''_{xx} = 12x^2 y \overset{(1,1)}{\rightsquigarrow} 12$

spojitá

$f''_{xy} = 4x^3 + 2y = f''_{yx} (= 4x^3 + 2y) \overset{(1,1)}{\rightsquigarrow} 6$

$f''_{yy} = 2x \overset{(1,1)}{\rightsquigarrow} 2$

$$\underset{(1,1)}{Tf}(x,y) = 5 + 6(x-1) + 3(y-1) + \frac{1}{2}\left(12(x-1)^2 + 2\cdot 6(x-1)(y-1) + 2(y-1)^2\right)$$

Pr2 $f(x,y) = \sqrt{x^2+y^2}$ $[x_0,y_0] = [3,4]$ $(x-x_0, y-y_0) = (-0{,}02,\ 0{,}05)$

$(x,y) = (2{,}98;\ 4{,}05)$ MINULE

$f'_x = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} \overset{(3,4)}{\rightsquigarrow} \frac{3}{5}$

$f'_y = \frac{y}{\sqrt{x^2+y^2}} \overset{(3,4)}{\rightsquigarrow} = \frac{4}{5}$

$f''_{xy} = \frac{-xy}{(\sqrt{x^2+y^2})^3} = f''_{yx} \overset{(3,4)}{\rightsquigarrow} \frac{-12}{5^3}$

$f''_{xx} = \frac{\sqrt{x^2+y^2} - \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}\cdot x}{x^2+y^2} = \frac{x^2+y^2-x^2}{(\sqrt{x^2+y^2})^3} = \frac{y^2}{(\sqrt{x^2+y^2})^3} = \frac{16}{5^3}$

$f''_{yy} = \frac{x^2}{(\sqrt{x^2+y^2})^3} = \frac{9}{5^3}$

$$\underset{(3,4)}{Tf}(2{,}98;\ 4{,}05) = 5 + \frac{3}{5}\cdot(-0{,}02) + \frac{4}{5}\cdot 0{,}05 + \frac{1}{2}\left[\frac{16}{5^3}\cdot 0{,}02^2 + 2\cdot\left(+\frac{12}{5^3}\right)0{,}02\cdot 0{,}05 + \frac{9}{5^3}\cdot 0{,}05^2\right] = 5{,}0282116$$

sme lepší o toto!

Pr3 $f(x,y) = x^3 + y^3 - 3xy$

NUTNÁ PODMIENKA pre extrém v (x_0,y_0):
ak existuje f'_x, f'_y na okolí (x_0,y_0) a $f'_x = 0$, $f'_y = 0$
t) STACIONÁRNY BOD

$f'_x = 3x^2 - 3y = 0 \Rightarrow x^2 = y \Rightarrow \boxed{y \geq 0}$

$f'_y = 3y^2 - 3x = 0 \Rightarrow y^2 = x \quad /^2$ $\boxed{x \geq 0}$

$y^4 = x^2 = y \Rightarrow y(y^3 - 1) = 0$

$y = 0, y = 1$

$x^2 = 0 \Rightarrow x = 0$

$x^2 = 1 \Rightarrow x = \pm 1$, preto $x = 1$!

Máme stacionárne body $\underset{P_1}{[0,0]}$, $\underset{P_2}{[1,1]}$

$f''_{xx} = 6x$

$f''_{xy} \overset{\text{spojité}}{=} f''_{yx} = -3$

$f''_{yy} = 6y$

$$Hf(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

$Hf(P_1) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ $\det Hf(P_1) < 0$ $\Rightarrow$ nie je extrém !

$Hf(P_2) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ $6 > 0$ $\det Hf = 36 - 9 > 0$

$\Rightarrow$ lok. minimum.

$Hf(P_1) = -6xy = -6(y' + x) \cdot x = -6x^2 - 6xy'$

$= \underbrace{-6}(x + \tfrac{1}{2}y')^2 + 6 \underbrace{\tfrac{1}{4}}(y')^2$

signatúra $(1, 1, 0)$ $\Rightarrow$ indefinitný

>0 >0
indefinitné

extrém nenastáva !

Z prednášky: $Hf(P_0)$

Ak $\det(H(f)) \neq 0$

- Hlavné minory $Hf(P)$ sú kladné $\Rightarrow P$ je lokálne minimum
  typický prípad $x^2 + y^2$
- Hlavné minory $Hf(P)$ sú $- + - +$ atď.
  $\Rightarrow P$ je lokálne maximum
  typický príklad $-(x^2 + y^2)$
- ~~Ak nenastáva žiadna z horných výsledkov~~ NIE JE EXTRÉM

Špeciálne pre $Hf(P) = \begin{pmatrix} f''_{xx}(P) & f''_{xy}(P) \\ f''_{yx}(P) & f''_{yy}(P) \end{pmatrix}$

Ak $\det Hf(P) > 0$ a $f''_{xx}(P) > 0$
$\Rightarrow$ lok. min.

Ak $\det Hf(P) > 0$ a $f''_{xx}(P) < 0$
$\Rightarrow$ lok. maximum.

symetrická !

Ak $\det Hf(P) < 0$, potom extrém v P nie je.

napr. pomocou vlastných čísel

$\det Hf(P) = \lambda_1 \cdot \lambda_2 < 0 \Rightarrow$ $\lambda_1 > 0, \lambda_2 < 0$ | $\lambda_1 < 0, \lambda_2 < 0$

INDEFINITNÁ.

④ $f(x,y) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$ $\boxed{x, y, z > 0}$

$f'_x = 1 - \frac{y^2}{4x^2} = 0 \Rightarrow 4x^2 = y^2 \Rightarrow \boxed{y = 2x}$

$f'_y = \frac{2y}{4x} - \frac{z^2}{y^2} = 0 \Rightarrow \frac{y}{2x} = \frac{z^2}{y^2}$

$1 = \frac{z^2}{y^2}$

$f'_z = \frac{2z}{y} - \frac{2}{z^2} = 0 \Rightarrow z^3 = y$

$\Rightarrow y^2 = z^2 \Rightarrow \boxed{y = z}$

$y(y^2-1) = y^3 - y = 0 \Rightarrow \boxed{y = 1}$ (vieme, že $y > 0$)

teda $z = y = 1$ a $x = \frac{y}{2} = \frac{1}{2}$ celkovo $P = [1/2, 1, 1]$

$\boxed{f''_{xx}} = 2\frac{y^2}{4x^3} \overset{(P)}{\rightsquigarrow} \boxed{4}$ $\boxed{f''_{yy}} = \frac{2}{4x} + 2y\frac{z^2}{y^3} \overset{P}{\rightsquigarrow} \boxed{3}$ $\boxed{f''_{zz}} = \frac{2}{y} + \frac{2 \cdot 2}{z^3} \overset{P}{\rightsquigarrow} \boxed{6}$

$\boxed{f''_{xy}} = -\frac{2y}{4x^2} = f''_{yx} \overset{P}{\rightsquigarrow} \boxed{-2}$ $\boxed{f''_{yz} = f''_{zy}} = -\frac{2z}{y^2} \overset{P}{\rightsquigarrow} \boxed{-2}$ $\boxed{f''_{xz} = 0}$

$$Hf(P) = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{pmatrix}$$

>0 | <0 | ① ② ③

4 | $3 - 4$ | $4 \cdot 3 \cdot 6 + 0 \cdot (-4) + 0 \cdot (-4) - (0 \cdot 3 + 4 \cdot 6 + 4 \cdot 4)$ ...

$\oplus$ $\ominus$ $\oplus$

SARRUSOVE PRAVIDLO

neg. definitná

$4 \cdot (18 - 6 - 4)$
$= 4 \cdot 8 = 32 > 0$

EXTRÉM NIE JE.

Pr 5 $f(x,y) = x^4 + y^4 - x^2 - 2xy - y^2 = x^4 + y^4 - (x+y)^2$

$f'_x = 4x^3 - 2(x+y) = 0$ $\ominus$

$f'_y = 4y^3 - 2(x+y) = 0$

$4x^3 - 4x = 0 = 4x(x^2-1)$

$\Rightarrow x = 0 \vee x = \pm 1$

pozor delenie/...

$0 = 4x^3 - 4y^3 = 4(x-y)(x^2+xy+y^2) \Rightarrow \boxed{x = y}$

Máme $P_1 = [0,0]$, $P_2 = [1,1]$, $P_3 = [-1,-1]$

$f''_{xx} = 12x^2 - 2$

$f''_{yy} = 12y^2 - 2$

$$Hf = \begin{pmatrix} 12x^2-2 & -2 \\ -2 & 12y^2-2 \end{pmatrix}$$

$f''_{xy} = -2 = f''_{yx}$

$Hf(P_1) = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$ lin. závislé

$\det Hf(P_1) = 0$ NEVIEME ROZHODNÚŤ.

$Hf(P_2) = \begin{pmatrix} 10 & -2 \\ -2 & 10 \end{pmatrix} = Hf(P_3)$

$\oplus$ $|100-4|$ $\oplus$ $\Rightarrow$ Lok. minimum.

Ukážeme, že v P_1 nie je lokálny extrém: $y = 0$ pre $x \in (-1,1)$

Zvolíme $x = -y$ $f(x,-x) = 2x^4 \geq 0$ a $f(x,0) = x^4 - x^2 = x^2(x^2-1) \leq 0$

nie je extrém, lebo $f(0,0) = 0$

Pr6 Určite lokálne extrémy funkcie $f(x,y) = xy\ln(x^2+y^2)$ na D_f.

$$f'_x = y\left(\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right) = 0 \Rightarrow y=0 \text{ alebo } (*)=0$$

symetricky:

$$f'_y = x\left(\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2}\right) = 0 \Rightarrow \cancel{x=0} \text{ alebo } (**)=0$$

1 2 | 3

$\boxed{D_f = \mathbb{R}^2 \setminus \{(0,0)\}}$!

1. $\boxed{y=0}$ dosadíme do $f'_y = 0$

máme $\boxed{\ln(x^2) = 0}$, pretože (0,0) $\notin D_f$ nemôže byť $x=0$

t.j. $x = \pm 1$ $\boxed{P_{1,2} = [\pm 1, 0]}$

2. $\boxed{x=0}$ symetrické k 1, teda $\ln(y^2)=0$ a $y=\pm 1$

$\boxed{P_{3,4} = [0, \pm 1]}$

(3)

$$\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = 0$$

$$\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2} = 0 \quad /(-1)$$

dosadíme napr. do 1. druhá bude platiť automaticky

$$\Rightarrow \frac{x^2-y^2}{x^2+y^2} = 0 \quad \text{t.j.} \quad \boxed{x^2=y^2} \quad \text{i.e. } x=\pm y$$

$$\ln(2x^2) + \frac{2x^2}{2x^2} = 0 \Rightarrow \boxed{\ln(2x^2) = -1} \Rightarrow x^2 = \frac{1}{2e}$$

$x = \pm\frac{1}{\sqrt{2e}}$ dostávame teda $P_{5,6} = \left[\pm\frac{1}{\sqrt{2e}}, \pm\frac{1}{\sqrt{2e}}\right]$

$P_{7,8} = \left[\pm\frac{1}{\sqrt{2e}}, \mp\frac{1}{\sqrt{2e}}\right]$

$(x,y) \in D_f$

$\boxed{1,2}$ $y=0 \vee x=0$, ak dosadíme do $f(x,y)$ vidíme, že

$f(x,0) = f(0,y) = 0$ a zrejme na okolí ~~týchto~~ bodov $f(x,y)$ nadobúda kladných aj záporných hodnôt $\Rightarrow$ extrém nie je v $P_{1,2}, P_{3,4}$

(3)

$$f''_{xx} = y\left(\frac{2x}{x^2+y^2} + \frac{4x(x^2+y^2) - 2x\cdot 2x^2}{(x^2+y^2)^2}\right) \overset{\boxed{y^2=x^2}}{=} y\left(\frac{2x}{2x^2} + \frac{4x\,2x^2 - 4x^3}{4x^4}\right) = \boxed{\frac{2y}{x}}$$

$$f''_{yy} = x\left(\frac{2y}{x^2+y^2} + \frac{4y(x^2+y^2) - 2y\,2y^2}{(x^2+y^2)^2}\right) \overset{\boxed{y^2=x^2}}{=} x\left(\frac{2y}{2y^2} + \frac{4y\,2y^2 - 4y^3}{4y^4}\right) = \boxed{\frac{2x}{y}}$$

$$f_{xy} = \ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} + y\left(\frac{2y}{x^2+y^2} + \frac{-2x^2\cdot 2y}{(x^2+y^2)^2}\right) \overset{\boxed{y^2=x^2}}{=} \underbrace{-1}_{\ln(2x^2)=-1} + \frac{2x^2}{2x^2} + \frac{2y^2}{2y^2} - \frac{4y^4}{4y^4} = \underline{\underline{0}}$$

$$Hf(P_{5-8}) = \begin{pmatrix} \frac{2x}{x} & 0 \\ 0 & \frac{2x}{y} \end{pmatrix} \qquad \det Hf(P_{5-8}) = 4-0 = 4 > 0$$

3. A $x = y$ $\quad f''_{xx}(P_{5,6}) = \frac{2y}{y} = 2 > 0$ = lokálne minimum

3. B $x = -y$ $\quad f''_{xx}(P_{7,8}) = \frac{2(-y)}{y} < 0$ = lokálne maximum.