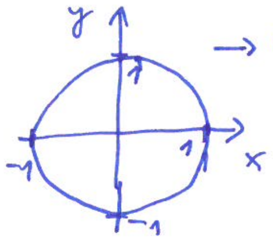
Pr 1 $\quad F(\underbrace{x,y}_{\text{nezávislé premenné}}) = x^2+y^2-1=0 \leftarrow$ zadáva táto rovnica na nejakej podmnožine $\mathbb{R}^2$ závislosť $y$ na $x$? $\quad$ CVIKO

| Implicitný zápis | Explicitný zápis |
|---|---|
| $x^2+y^2-1=0,$ $y>0$ | $y=\sqrt{1-x^2}$ |




→ vidíme, že až sa obmedzíme na hornú polkružnicu ($y>0$) alebo spodnú polkružnicu ($y<0$) získame funkčný vzťah $y=f(x)$.

1. Derivácia implicitne zadanej funkcie

vzorec z prednášky

$$2x + 2\cdot y y'(x) = 0 \Rightarrow y'(x) = -\frac{x}{y} = \boxed{-\frac{F'_x}{F'_y}}$$

Cez explicitný popis je to zbytočnejšie → $y'(x) = \frac{1}{2}\frac{1}{\pm\sqrt{1-x^2}}\cdot(-2x) = \frac{-x}{y}$

2. Derivácia $\quad 2 + 2(y'(x))^2 + 2y\cdot y''(x) = 0$

Extrémy funkcie: $\quad y'(x) = -\frac{x}{y} = 0 \Rightarrow \boxed{x=0}$

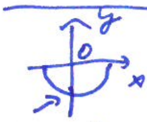
$$2 + 2\cdot 0 + 2\cdot y\, y''(0) = 0 \Rightarrow y''(0) = -\frac{1}{y}$$

| | |
|---|---|
| $-1<0$ | $y>0$ |
| $1>0$ | $y<0$ |

Lok. MAX


0 je bod lokálneho maxima pre $x^2+y^2-1=0 \wedge y>0$

0 je bod lokálneho minima pre $x^2+y^2-1=0 \wedge y<0$




Lok. MIN

(2) $\quad xy^2 - 2xy + x^3 - 3y^2 + 5 = 0 \quad \boxed{y=y(x)}$

$$y^2 + 2xyy' - 2y - 2xy' + 3x^2 - 6yy' = 0$$
$$y^2 - 2y + 3x^2 = y'(-2xy + 2x + 6y)$$

$$y' = \frac{y^2-2y+3x^2}{-2xy+2x+6y \neq 0} \text{ až}$$

(3) $\quad \sin(x^2) + \cos(y^2) - 1 = 0$

$$\cos(x^2)2x + (-\sin(y^2)\cdot 2yy') = 0$$
$$\cos(x^2)\cdot 2x = \sin(y^2)\cdot 2y\cdot y'$$

$$y' = \frac{\cos(x^2)\,x}{\sin(y^2)\cdot y} \quad \text{až } \neq 0$$

(4) $\quad \underline{3y^2 y'} - 2y - \underline{2xy'} + 2x = 0 \qquad y' = \frac{-2y-2x}{-3y^2+2x} \to y'(1) = \frac{-2+2}{-3+2} = 0.$

(4) $\underbrace{-2y'}_{0} + 2 = -\underbrace{3\cdot 2\cdot y(y')^2}_{=0} - 3y^2\cdot y'' + \underbrace{2y'}_{=0} + 2xy'' \quad /(1)$

$2 = -3\cdot 1\cdot y''(1) + 2\cdot 1\cdot y''(1) \Rightarrow y''(1) = -2$

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(5) $x^3 - y^3 + 2xy = 0$

$\boxed{\text{cviko}}$

$3x^2 - 3y^2\cdot y' + 2y + 2xy' = 0$ $\qquad y' = \frac{3x^2+2y}{3y^2-2x}$

$y'(1) = \frac{3\cdot 1-2}{3\cdot 1+2} = \qquad = 1$

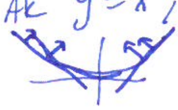
$3\cdot 2\cdot x - 6y\cdot(y')^2 - 3y^2\cdot y'' + 2y' + 2y' + 2y'' = 0$

$\boxed{x=1,\ y=-1}$

$6\cdot 1 + 6\cdot 1\cdot 1 - 3\cdot 1\cdot y''(1) + 2\cdot 2 + 2y''(1) = 0$

$\boxed{y''(1) = 16} > 0$ | Ak $y=x^2$, potom $y''=2>0$

konvexná



– konvexná

tečna v bode $[1,-1]$ leží pod grafom funkcie

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(6) $\frac{9}{2}x^2 - 3xy^2 + y^3 - \frac{9}{2} = 0$

$\frac{9}{2}\cdot 2x - 3y^2 - 3x\cdot 2yy' + 3y^2y' = 0$

$y' = \frac{9x-3y^2}{6xy-3y^2}$ $\qquad y'(1) = \frac{9-3\cdot 3^2}{6\cdot 3-3\cdot 3^2} = \frac{-2}{-1}$

$\boxed{y'(1) = 2}$

$9 - 6yy' - 6yy' - 6x(y')^2 - 6xyy'' + 6y(y')^2 + 3y^2y'' = 0$ / dosadíme 1

$0 = 9 - 6\cdot 3\cdot y'(1) - 6\cdot 3y'(1) - 6\cdot 1(y'(1))^2 - 6\cdot 1\cdot 3y''(1) + 6\cdot 3\cdot (y'(1))^2 + 3\cdot 3^2 y''(1)$

$9 - 18\cdot 2 - 18\cdot 2 - 6\cdot 4 - 18y''(1) + 18\cdot 4 + 3^3\cdot y''(1) = 0$

$-15 + 9y''(1) = 0$

$y''(1) = \frac{15}{9} > 0$

konvexná, graf leží nad tečnou v bode $[1,3]$

(7) bod $[1,\sqrt{2},2]$, $z = f(x,y)$ $\qquad x^2+y^2+z^2-xz-\sqrt{2}yz = 1$

$2x + 2zz'_x - z - xz'_x - \sqrt{2}yz'_x = 0 \Rightarrow z'_x = \frac{2x-z}{-2z+x+\sqrt{2}y} \overset{[1,\sqrt{2},2]}{=} \frac{0}{-1}$

$2y + 2zz'_y - xz'_y - \sqrt{2}z - \sqrt{2}yz'_y = 0 \Rightarrow z'_y = \frac{2y-\sqrt{2}z}{-2z+x+\sqrt{2}y} \overset{[1,\sqrt{2},2]}{=} \frac{0}{-4+1+2}$

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$2 + 2\underbrace{(z'_x)^2}_{0} + 2zz''_{xx} - \underbrace{z'_x}_{0} - \underbrace{z'_x}_{0} - xz''_{xx} - \sqrt{2}yz''_{xx} = 0 \Rightarrow z''_{xx} = \frac{+2}{-2z+x+\sqrt{2}y} = \frac{+2}{-1}$

$2 + 2\underbrace{(z'_y)^2}_{0} + 2zz''_{yy} - xz''_{yy} - \sqrt{2}\underbrace{z'_y}_{0} - \sqrt{2}\underbrace{z'_y}_{0} - \sqrt{2}yz''_{yy} = 0 \Rightarrow z''_{yy} = \frac{2}{-2z+x+\sqrt{2}y} = -2$

$2\underbrace{z'_yz'_x}_{0} + 2zz''_{xy} - \underbrace{z'_y}_{0} - xz''_{xy} - \sqrt{2}\underbrace{z'_x}_{0} - \sqrt{2}yz''_{xy} = 0 \Rightarrow z''_{xy} = \frac{0}{-2z+x+\sqrt{2}y} = 0$

$\textcircled{8}$ $\boxed{P \overset{\text{bod}}{=} [-2,0,1]\ ,\ z = f(x,y)}$ $\quad 2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0$ $\quad \boxed{\text{CVIKO}}$

$z'_x(P) = \dfrac{4x+8z}{-2z-8x+1} = 0 \Leftarrow \quad \bullet\ 4x + \underline{2zz'_x} + 8z + \underline{8xz'_x} - \underline{z'_x} = 0$

$z'_y(P) = \dfrac{4y}{-2z-8x+1} = 0 \Leftarrow \quad \bullet\ 4y + \underline{2zz'_y} + \underline{8xz'_y} - \underline{z'_y} = 0$

$z''_{xx}(P) = \dfrac{-4}{2z+8x-1} = \dfrac{4}{15} \Leftarrow \quad \bullet\ 4 + \underbrace{2(z'_x)^2}_{=0} + \underline{2zz''_{xx}} + \underbrace{8z'_x}_{=0} + \underbrace{8z'_x}_{=0} + \underline{8xz''_{xx}} - \underline{z''_{xx}} = 0$

$z''_{yy}(P) = \dfrac{-4}{2z+8x-1} = \dfrac{4}{15} \Leftarrow \quad \bullet\ 4 + \underbrace{2(z'_y)^2}_{=0} + \underline{2zz''_{yy}} + \underline{8xz''_{yy}} - \underline{z''_{yy}} = 0$

$z''_{xy}(P) = 0 \quad \Leftarrow \quad \bullet\ \underbrace{2z'_y z'_x}_{=0} + 2z''_{xy} + \underbrace{8z'_y}_{=0} + 8xz''_{xy} - z''_{xy} = 0$

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$\textcircled{9}$ $F(x,y) = x^2 + 2xy - y^2 - 8 = 0$ $\quad \boxed{\text{cviko}}$ dosadíme

Podozrivé body sú: $F'_y(x,y) = 0$

$F'_y = 2x - 2y = 0 \Rightarrow x = y$

$\cancel{x^2} + 2x^2 - \cancel{x^2} - 8 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$[\pm 2, \pm 2]$

KRATŠÍ SPÔSOB výpočtu:

$F(x, f(x)) = 0$

$F'_x + F'_y \cdot f'(x) = 0$

$\Rightarrow f'(x) = -\dfrac{F'_x(x,y)}{F'_y(x,y)}$

ak $F'_y(x,y) \neq 0$ a $\boxed{y = f(x)}$

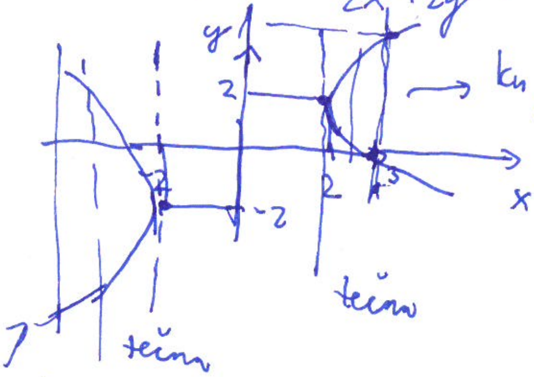
$x = x(y)$ (hodnoty x závisia na y) Podľa vzorečka máme: $x'(y) = \dfrac{-F'_y}{F'_x}$

$F'_x = 2x + 2y$

$x'(y) = -\dfrac{(2x-2y)}{2x+2y} = -\dfrac{x-y}{x+y}$

$\boxed{x'(2) = 0 = x'(-2)}$ v bode $\pm 2$

smernica tečny je rovnobežná s osou y




ku každej hodnote x existujú 2 hodnoty y $\Rightarrow$ nie je funkcia

TEČNA: $x - x(\pm 2) = \overset{=0}{x'(\pm 2)}(x \mp 2)$ $\Rightarrow \boxed{x = \pm 2}$

$9 + 6y - y^2 - 8 = 0 \Leftrightarrow -y^2 + 6y + 1 = 0$

$D = 36 + 4 = 40 \quad \overset{2\sqrt{10}}{=}$

$x(0) \approx x(6) \approx 3 \Leftarrow \quad y_1, y_2 = \dfrac{-6 \pm \sqrt{40}}{-2} \approx \dfrac{3 \mp \sqrt{10}}{0,6}$

je chcozmožné priradiť k x jeho funkčnú hodnotu $\Rightarrow$ odpoveď je $\underline{[\pm 2, \pm 2]}$

$(x - y) + (x+y)\,x'(y) = 0$

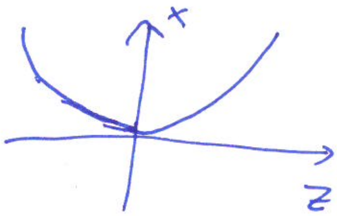
$1 + \underbrace{x'(y)}_{=0} + (x+y)\,x''(y) = 0 \qquad x''(\pm 2) = \left.\dfrac{-1}{x+y}\right|_{\substack{x=\pm 2\\ y=\pm 2}} = \begin{cases} -\frac{1}{4} & x=2 \quad \text{LOK. MAX} \\ \frac{1}{4} & x=-2 \quad \text{LOK. MIN.} \end{cases}$

KRIVKA LEŽÍ NAD TEČNOU ($x=-2$) / POD TEČNOU ($x=2$)

$z^2 - 2px = 0$ , $p > 0$ $\quad z = f(x,y)$ (10)

$F'_z = 2z = 0 \Rightarrow z = 0$ podozrivé body sú $z=0$, $0 - 2px = 0 \Rightarrow x = 0$

$[0, y, 0]$ $\underline{y \in \mathbb{R}}$



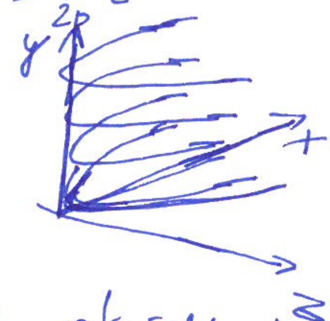


Graf funkcie $x = g(z,y)$

(1) rez rovinou $y = c \in \mathbb{R}$ pevné

$z^2 = 2px \Rightarrow x = \frac{1}{2p} z^2$

(2) Samotný graf:





Vidíme, že každé okolie $O$ bodu $[0, c, 0]$ obsahuje pre každé $[a, c, b] \in O(\ldots)$ $a, b > 0$ aj bod $[a, c, -b] \in O$.

Teda nie je možné vyjadriť $z$ v závislosti na $x$ na okolí bodu $[0, c, 0]$ $\forall c \in \mathbb{R}$.

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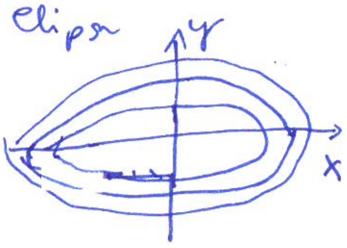
(11) $F'_z = -\frac{2z}{c^2} = 0 \Rightarrow z = 0$ Podozrivé body sú body elipsy t.j.

$\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$

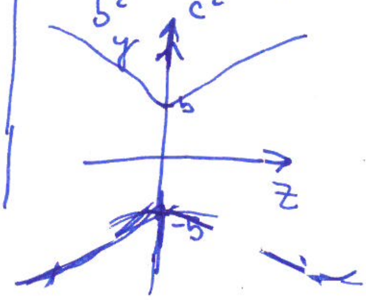
rez rovinami $z = c$ elipsa





$\boxed{x = 0}$ $\quad y = \pm b\sqrt{1 + \frac{z^2}{c^2}}$

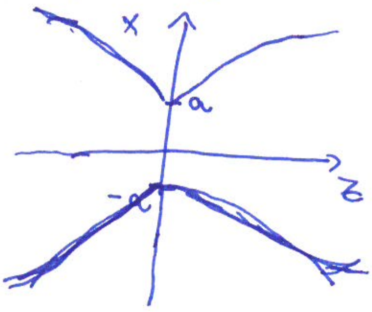
$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$





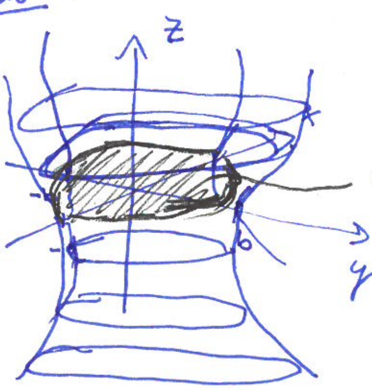
$\boxed{y = 0}$ $\quad x = \pm a\sqrt{1 + \frac{z^2}{c^2}}$

$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$





Graf:





elipsa pre $z = 0$

$\rightarrow$ „presýpacie hodiny"

zase: v každom okolí ľubovoľného bodu elipsy ($z = 0$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$). leží pre bod $[x, y, w]$ bod $[x, y, -w]$ pre $w > 0$.

$\Rightarrow$ Nie je možné vyjadriť $z = f(x,y)$ na okolí bodu elipsy

⑧ Prvá derivácia cez matice: $F: \mathbb{R}^{k+n} \to \mathbb{R}^n$, $x \in \mathbb{R}^k$, $y \in \mathbb{R}^n$, $y = G(x)$

Vzorec z prednášky $D^1(G(x)) = -(D^1_y F)^{-1}(x, G(x))\, D^1_x F(x, G(x))$

Pre nás je $\boxed{n=1,\ k=2}$

$D^1_y F = F'_z = 2z + 8x - 1$ $\boxed{y = z}$ $F(x,y,z) = 2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0$

$D^1_x F = (F'_x, F'_y) = (4x + 8z,\ 4y)$

$\boxed{x = (x,y)}$ $D^1(z(x,y)) = \dfrac{-1}{2z + 8x - 1} \cdot (4x + 8z,\ 4y)$ t.j. $z'_x = -\dfrac{4x + 8z}{2z + 8x - 1}$

$z'_y = -\dfrac{4y}{2z + 8x - 1}$.

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