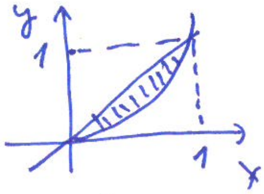
Pr 10

S: $y = x$
$y = x^2$



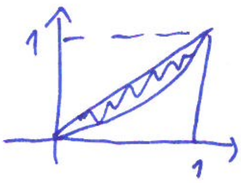
$$\iint_S xy^2\,dx\,dy$$

$$\int_0^1\int_{x^2}^{x} xy^2\,dy\,dx = \int_0^1 \left[x\frac{y^3}{3}\right]_{x^2}^{x} dx = \frac{1}{3}\int_0^1 x^4 - x^7\,dx = \frac{1}{3}\left[\frac{x^5}{5}-\frac{x^8}{8}\right]_0^1$$

$$= \frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right) = \frac{8-5}{40}\cdot\frac{1}{3} = \frac{1}{40}$$

Pr 11

A: $y = x$
$y = x^3$



$$\iint_A x^3y\,dx\,dy$$

$$\int_0^1\int_{x^3}^{x} x^3y\,dy\,dx = \frac{1}{2}\int_0^1 x^3\left[y^2\right]_{x^3}^{x} dx = \frac{1}{2}\int_0^1 x^5 - x^9\,dx = \frac{1}{2}\left[\frac{x^6}{6}-\frac{x^{10}}{10}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{6}-\frac{1}{10}\right) = \frac{1}{2}\cdot\frac{5-3}{2\cdot 3\cdot 5} = \frac{1}{30}$$

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CVIČENIE 6

chceme aby bolo invertibilné

$G(x,y): M \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, prvky Jacobiho matica $G'(x,y)$, sú spojité funkcie

$\det G'(x,y) \neq 0 \quad \forall (x,y) \in M$

existuje inverzia k $G'(x,y)$

$$\iint\limits_{G(K)} f(s,t)\, \overset{ds \wedge dt}{ds \cdot dt} = \iint\limits_{K} f(G(x,y)) \cdot |\det \underbrace{G'(x,y)}_{J}|\, \underset{dx \wedge dy}{dx dy}$$

$dx \wedge dy = -dy \wedge dx$

$\boxed{ds \wedge ds = 0}$

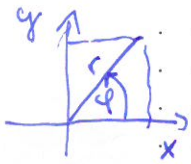
$(s,t) = G(x,y) \qquad (s,t) = (f(x,y), g(x,y))$

$d(f(x,y))$

reťazcové pravidlo $\to$ $ds = f'_x(x,y) \cdot dx + f'_y \cdot dy$

$dt = g'_x(x,y) \cdot dx + g'_y(x,y) dy$

$$\begin{pmatrix} ds \\ dt \end{pmatrix} = \begin{pmatrix} f'_x(x,y) & f'_y(x,y) \\ g'_x(x,y) & g'_y(x,y) \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$



napr. $x = r \cdot \cos\varphi + a$

$y = r \cdot \sin\varphi + b$

$r \in (0, \infty)$, $\varphi \in [0, 2\pi)$

POLÁRNE SÚRADNICE

$dx = \cos\varphi\, dr - r \cdot \sin\varphi\, d\varphi$

$dy = \sin\varphi\, dr + r \cdot \cos\varphi\, d\varphi$

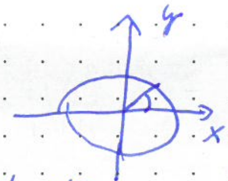
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}}_{J} \begin{pmatrix} dr \\ d\varphi \end{pmatrix}$$

TRANSFORMÁCIU URČUJEME PODĽA INTEGROVANEJ OBLASTI A NIE FUNKCIE, CEZ KTORÚ INTEGRUJEME

CVIKO

Pr1

$I = \iint_M f(\sqrt{x^2+y^2})\, dx dy \qquad M: x^2+y^2 \leq 1$



$\varphi \in [0, 2\pi)$

$r \in [0, 1)$

$$I = \int_0^{2\pi}\int_0^1 f(r) \cdot r\, dr d\varphi = 2\pi \int_0^1 f(r) \cdot r\, dr d\varphi$$

chceme sa zbaviť „posunutia"

Pr2

$$\iint_M \sqrt{(x-1)^2+(y+1)^2}\, dx dy$$

$M: 1 \leq (x-1)^2+(y+1)^2 \leq 4$

$1^2 \qquad S[1,-1] \qquad 2^2$

$x = r \cdot \cos\varphi + 1$

$y = r \cdot \sin\varphi - 1$

$\varphi \in [0, 2\pi)$

$r \in [1,2]$ lebo $1 \leq (r\cos\varphi + 1 - 1)^2 + (r\sin\varphi - 1 + 1)^2 \leq 2^2$

$1 \leq r^2 \leq 2^2$

$$\int_0^{2\pi}\int_1^2 \sqrt{r^2} \cdot r \cdot dr d\varphi = \int_0^{2\pi} \left[\frac{r^3}{3}\right]_1^2 d\varphi = \frac{7}{3} \cdot 2\pi$$

CVIKO

Pr3

$\boxed{\begin{matrix} u = x \cdot y \\ y = v x \end{matrix}}$

$$I = \iint_A xy^2\, dx dy$$

A: $\begin{matrix} xy = 1/2 \\ x \cdot y = 2 \end{matrix}$ | $\begin{matrix} v = \frac{1}{2} \\ 2y = x \\ y = 2x \\ v = 2 \end{matrix}$ $\quad x, y \geq 0$

$\boxed{\frac{1}{2} \leq u \leq 2, \quad \frac{1}{2} \leq v \leq 2}$

vyjadrím $v = \frac{y}{x} \to$ $\boxed{\text{v našej oblasti } x \neq 0}$

$$(du, dv) = \underbrace{\begin{pmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}}_{J} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$u'_x = y, \; u'_y = x$

$v'_y = \frac{1}{x}, \; v'_x = -\frac{y}{x^2}$

Potrebujeme $\begin{pmatrix} dx \\ dy \end{pmatrix} = J^{-1}\begin{pmatrix} du \\ dv \end{pmatrix}$ a vieme, že $\det(J^{-1}) = \dfrac{1}{\det(J)}$ ak $\neq 0$

$\frac{1}{2} \le u \le 2$, $\frac{1}{2} \le v \le 2$ $\quad x^2y^2 = (xy)^2 = u^2$

$\det(J) = \frac{y}{x} + \frac{y}{x} = 2\frac{y}{x} = 2v$

$$I = \int_{1/2}^{2}\int_{1/2}^{2} u^2 \cdot |\det J^{-1}|\, du\, dv = \int_{1/2}^{2}\int_{1/2}^{2} u^2 \cdot \frac{1}{2v}\, du\, dv = \frac{1}{2}\int_{1/2}^{2} \left[\frac{u^3}{3}\right]_{1/2}^{2} \frac{1}{v}\, dv =$$

$$= \frac{1}{2}\frac{1}{3}\left(8 - \frac{1}{8}\right)\left[\ln|v|\right]_{1/2}^{2} = \frac{1}{6}\,\frac{63}{8}\left(\ln 2 + \ln 2\right) = \frac{63}{24}\ln 2$$

Iný prístup (zložitejší) $\quad u = x\cdot y$, $y = v\cdot x \Rightarrow u = x\cdot v\cdot x \Rightarrow x = \sqrt{u\cdot v}$

$v = \frac{y}{x}$ $\quad y = v\cdot x = v\cdot\sqrt{u\cdot v}$

a počítame rovno $\boxed{\begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} x'_u & x'_v \\ y'_u & y'_v \end{pmatrix}}_{K}\begin{pmatrix} du \\ dv \end{pmatrix}}$

dostaneme hneď $\det K = \frac{1}{2v}$

$\det(I) = 1$

$\det(J^{-1}J) = \det(J^{-1})\cdot\det(J)$

Pr4 $\quad u = xy$, $v = \frac{y^2}{x}$ $\qquad I = \iint_A \sqrt{xy}\, dx\, dy$

A: $\left.\begin{array}{l} y^2 = 2x \\ y^2 = x \end{array}\right\} v \in [1,2]$

$\left.\begin{array}{l} u = xy = 1 \\ u = xy = 2 \end{array}\right\} u \in [1,2]$

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \underbrace{\begin{pmatrix} y & x \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{pmatrix}}_{J}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

$|\det J| = \left|\frac{2y^2}{x} + \frac{y^2}{x}\right| = \underline{\underline{3v}}$

$|\det J^{-1}| = \frac{1}{3v}$

$$\int_1^2\int_1^2 \sqrt{u}\,\frac{1}{3v}\cdot du\, dv = \frac{1}{3}\int_1^2 \left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^2 \cdot \frac{1}{v}\, dv = \frac{1}{3}\cdot\frac{2}{3}\cdot\left(2^{3/2} - 1\right)\left[\ln|v|\right]_1^2 =$$

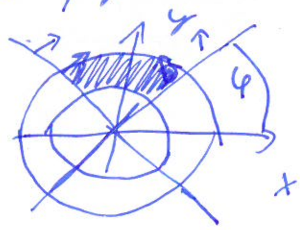
$$\frac{2}{9}\cdot\left(2\sqrt{2} - 1\right)\cdot\ln 2$$

Pr5 $\quad \iint_A 2(x^2+y^2)\, dA$ $\qquad A$: $1 \le x^2+y^2 \le 4$, $y \ge |x|$ $\qquad \varphi = \frac{\pi}{4}$ (45°)

$1 \le r^2 \le 2^2$

$r \in [1,2]$

$\varphi \in \left[\frac{\pi}{4}, \frac{3}{4}\pi\right]$



$$= \int_1^2\int_{\frac{\pi}{4}}^{3/4\pi} 2r^2\cdot r\, d\varphi\, dr$$

$$= 2\cdot\frac{\pi}{2}\cdot\left[\frac{r^4}{4}\right]_1^2 = \frac{\pi}{4}\cdot(16-1) = \frac{15}{4}\pi$$

CVIKo

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obsah plochy

$0 \le x \le 1$

$-\sqrt{x-x^2} \le y \le \sqrt{1-x^2}$

hranice: $y = \sqrt{1-x^2}$

$y^2 = 1-x^2$

horná polkružnica

Pr 6 $I = \int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{1-x^2}} 1\, dy\, dx = \boxed{\underbrace{\int_0^1 \int_0^{\sqrt{1-x^2}} 1\, dy\, dx}_{I_1} + \underbrace{\int_0^1 \int_{-\sqrt{x-x^2}}^{0} 1\, dy\, dx}_{I_2}}$

$y = -\sqrt{x-x^2}$ → spodná polkružnica posunutá

$y^2 = x - x^2$

$x^2 - x + y^2 = 0$

$x^2 - 2\cdot\frac{1}{2}x + y^2 = 0$

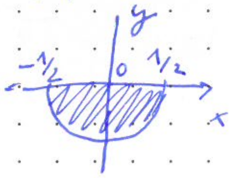
$\left(x-\frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$

$I_1 = \int_0^1 \int_0^{\sqrt{1-x^2}} 1\, dy\, dx \;\left|\; \begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ r \in (0,1) \\ \varphi \in [0,\pi/2] \\ |J| = r \end{array} \right| = \int_0^1 \int_0^{\pi/2} r\, d\varphi\, dr = \frac{\pi}{2}\cdot\left[\frac{r^2}{2}\right]_0^1 = \frac{\pi}{4}$

(plocha štvrť kruhu t.j. $\frac{\pi}{4} r^2 = \frac{\pi}{4}$)

$I_2 = \int_0^1 \int_{-\sqrt{x-x^2}}^{0} 1\, dy\, dx = \left|\; \begin{array}{l} x = r\cos\varphi - 1/2 \\ y = r\sin\varphi \\ r \in (0, 1/2) \\ \varphi \in (+\pi, +2\pi) \\ |J| = r \end{array} \right| \int_0^{1/2} \int_{+\pi}^{+2\pi} r\, d\varphi\, dr = \pi\left[\frac{r^2}{2}\right]_0^{1/2} = \frac{1}{8}\pi$

(plocha polkruhu o polomere $\frac{1}{2}$: $\frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{1}{8}\pi$)



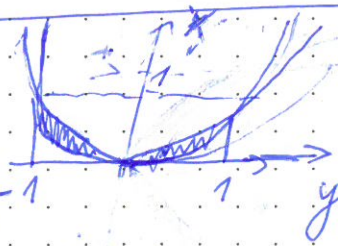
$I = I_1 + I_2 = \underline{\underline{\frac{\pi}{4} + \frac{1}{8}\pi}}$

Pr 7 $S = \iint_A dx\,dy = 2\cdot I$

$A: x = y^2,\ x = 4y^2 - 3$

$4y^2 - 3 = x = y^2$

$3y^2 - 3 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$



$I = \int_0^1 \int_{4y^2-3}^{y^2} dx\,dy$

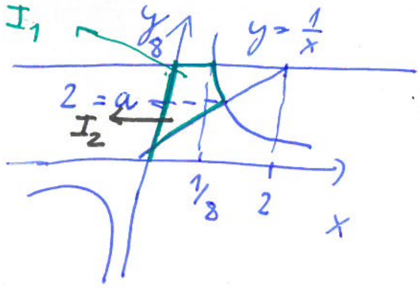
zrejme $y^2 \geq 4y^2 - 3$ na $y \in \langle 0,1\rangle$

$= \int_0^1 y^2 - 4y^2 + 3\, dy = \left[-\frac{3y^3}{3} + 3y\right]_0^1 = 3 - 1 = \underline{\underline{2}}$

výsledok $\underline{\underline{2\cdot 2 = 4}}$

Pr8 $A: x=0, y=\frac{1}{x}, y=8, y=4x$

$I=\iint_A dx\,dy = I_1 + I_2$



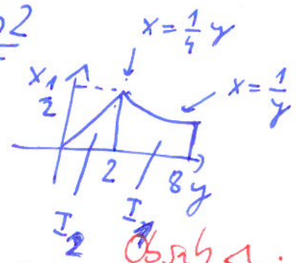
$a: \frac{1}{x}=4x$

$1=4x^2$

$\boxed{x=\frac{1}{2}}$

$$I_1=\int_2^8\int_0^{1/y} dx\,dy = \int_2^8 \frac{1}{y}\,dy = [\ln|y|]_2^8 = 3\ln 2-\ln 2 = \underline{2\ln 2}$$

$$I_2=\int_0^2\int_0^{\frac{1}{4}y} 1\,dx\,dy = \int_0^2 \frac{1}{4}y\,dy = \frac{1}{4}\left[\frac{y^2}{2}\right]_0^2 = \frac{1}{2}$$



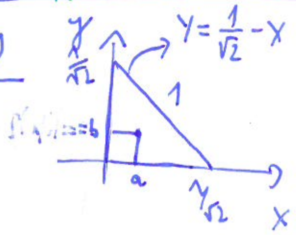
Obsah Δ:

$I_2 = 2\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$

$I=\frac{1}{2}+2\ln 2$

$\boxed{\text{CvIko}}$

Pr9



$x\in[0,1/\sqrt{2}]$

$0\le y\le \frac{1}{\sqrt{2}}-x$

$\rho(a,b)=b\cdot k$ špeciálne $\rho(0,\frac{1}{\sqrt{2}})=2=\frac{1}{\sqrt{2}}\cdot k \Rightarrow k=2\cdot\sqrt{2}$

$=2\cdot\sqrt{2}\cdot b$

$$M=\int_0^{1/\sqrt{2}}\int_0^{\frac{1}{\sqrt{2}}-x} 2\cdot\sqrt{2}\cdot y\,dy\,dx = 2\cdot\sqrt{2}\int_0^{1/\sqrt{2}}\left[\frac{y^2}{2}\right]_0^{1/\sqrt{2}-x}dx$$

$$=\sqrt{2}\cdot\int_0^{1/\sqrt{2}}(1/\sqrt{2}-x)^2\,dx = -\sqrt{2}\left[\frac{(1/\sqrt{2}-x)^3}{3}\right]_0^{1/\sqrt{2}}$$

$$=\frac{\sqrt{2}}{3}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{1}{6}$$

$$x_0=\frac{1}{M}\int_0^{1/\sqrt{2}}\int_0^{1/\sqrt{2}-x} 2\sqrt{2}y\cdot x\,dy\,dx = 2\sqrt{2}\cdot 6\int_0^{1/\sqrt{2}}\left[\frac{y^2}{2}\right]_0^{1/\sqrt{2}-x}\cdot x\,dx = 6\sqrt{2}\int_0^{1/\sqrt{2}}\left(\frac{1}{\sqrt{2}}-x\right)^2\cdot x\,dx$$

$$=6\sqrt{2}\left(\left[-\frac{(1/\sqrt{2}-x)^3}{3}\cdot x\right]_0^{1/\sqrt{2}}-\int_0^{1/\sqrt{2}}-\frac{(1/\sqrt{2}-x)^3}{3}\,dx\right)=$$

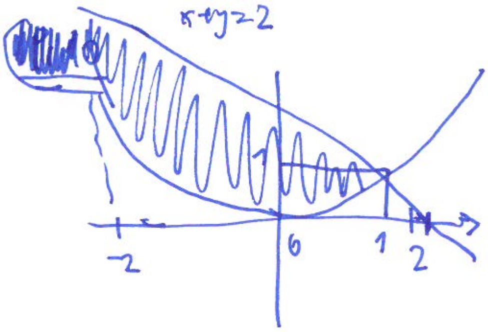
$$=2\sqrt{2}\cdot-\left[\frac{(1/\sqrt{2}-x)^4}{4}\right]_0^{1/\sqrt{2}}=\frac{2\sqrt{2}}{16}=\frac{\sqrt{2}}{8}$$

$$y_0=\frac{1}{M}\int_0^{1/\sqrt{2}}\int_0^{1/\sqrt{2}-x}2\sqrt{2}\cdot y^2\,dy\,dx = 6\cdot 2\cdot\sqrt{2}\,\frac{1}{3}\int_0^{1/\sqrt{2}}\left[\left(\frac{1}{\sqrt{2}}-x\right)\right]^3dx = -4\sqrt{2}\left[\frac{(\frac{1}{\sqrt{2}}-x)^4}{4}\right]_0^{1/\sqrt{2}}$$

$=\frac{\sqrt{2}}{\ }$

$\boxed{\text{CvIko}}$

Pr10



$y = x^2$

$x + y = 2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$$M = \int_{-2}^{1}\int_{x^2}^{2-x} dy\,dx = \int_{-2}^{1} 2 - x - x^2\,dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= 2 - \tfrac{1}{2} - \tfrac{1}{3} + 4 + 2 - \frac{8}{3}$$

$$= 8 - 3 - \tfrac{1}{2} = 5 - \tfrac{1}{2} = \frac{9}{2}$$

Pr 10

Cvičko

$$x_0 = \frac{1}{M}\int_{-2}^{1}\int_{x^2}^{2-x} x\,dy\,dx = \frac{1}{M}\int_{-2}^{1} x(2 - x - x^2)\,dx = \frac{1}{M}\int_{-2}^{1} 2x - x^2 - x^3\,dx$$

$$= \frac{1}{M}\left[2\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right]_{-2}^{1} = \frac{1}{M}\left(1 - \frac{1}{3} - \frac{1}{4} - \cancel{4} - \frac{8}{3} + \cancel{4}\right)$$

$$= \frac{1}{M}\left(-2 - \frac{1}{4}\right) = \frac{2}{9}\cdot\frac{-9}{4} = -\frac{1}{2}$$

$$y_0 = \frac{1}{M}\int_{-2}^{1}\int_{x^2}^{2-x} y\,dy\,dx = \frac{1}{2M}\int_{-2}^{1}\left[y^2\right]_{x^2}^{2-x} dx = \frac{1}{2M}\int_{-2}^{1}(2-x)^2 - x^4\,dx$$

$$= \frac{1}{2M}\left[-\frac{(2-x)^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{1}{9}\left(-\frac{1}{3} - \frac{1}{5} + \frac{4^3}{3} - \frac{32}{5}\right) = \frac{1}{9}\left(-\frac{33}{5} + \frac{63}{3}\right)$$

$$= \frac{1}{9}\left(-\frac{33}{5} + 21\right) = \frac{1}{9}\,\frac{-33 + 105}{5} = \frac{72}{45}$$

$$= \frac{8}{5}$$